ON THE CODIMENSION GROWTH OF G-GRADED ALGEBRAS

ELI ALJADEFF

ABSTRACT. Let W be an associative PI affine algebra over a field F of characteristic zero. Suppose W is G-graded where G is a finite group. Let $\exp(W)$ and $\exp(W_e)$ denote the codimension growth of W and of the identity component W_e, respectively. We prove: $\exp(W) \leq |G|^2 \exp(W_e)$. This inequality had been conjectured by Bahturin and Zaicev.

1. Introduction

Let W be a PI-affine algebra over a field F of characteristic zero. The codimension growth of W was studied by several authors (see for instance [4], [5], [7], [8], [9], [10], [11], [14], [16]). It provides an important tool which measures the "size" of the T-ideal of identities of W in asymptotic terms. In case the algebra W is G-graded, it is natural to compare the codimension growths of W and W_e, where W_e is the identity component of W with respect to the given G-grading.

Let us recall briefly the definitions. Let X be a countable set of indeterminates and $F\langle X\rangle$ the corresponding free algebra over F. Let $\mathrm{id}(W)$ be the T-ideal of identities of W in $F\langle X\rangle$ and let $\mathcal{W} = F\langle X\rangle / \mathrm{id}(W)$ denote the relatively free algebra of W. We denote by $c_n(W)$ the dimension of the subspace spanned by multilinear elements in n free generators in $\mathcal{W}$. Our interest is in the asymptotic behavior of the sequence of codimensions, namely in

$$\exp(W) = \lim_{n\to\infty} \sqrt[n]{c_n(W)}.$$

It is known that $\exp(W)$ exists and moreover it assumes only integer values (see [9], [10]).

Now suppose that the algebra W is G-graded, where G is an arbitrary finite group. Clearly, the identity component W_e of W, is a subalgebra and we may consider $\exp(W_e)$.

In [4] Y. A. Bahturin and M. V. Zaicev made the following conjecture.

Conjecture 1.1.

$$\exp(W) \leq |G|^2 \exp(W_e).$$

The conjecture was proved under certain natural conditions imposed on the grading group G and on the algebra W (see [4]):

(1) The algebra W is finite-dimensional over a field of characteristic zero and W_e has polynomial growth.

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(2) The algebra W is finitely generated semisimple over a field of characteristic zero and G is abelian.

Our goal in this article is to prove the conjecture for affine algebras where G is an arbitrary finite group. For future reference we record this in

Theorem 1.2. *Let G be a finite group. If W is an PI-affine G-graded algebra over a field F of characteristic zero then* $\exp(W) \leq |G|^2 \exp(W_e)$.

The first step (which is a key step) is to reduce the problem to finite-dimensional G-graded algebras. This is obtained by invoking the "Representability Theorem" for G-graded affine algebras (see [2], Theorem 1.1). In order to state the theorem recall that for a G-graded algebra one can define in a natural way G-graded identities and can consider the T-ideal of G-graded identities $\mathrm{id}_G(W)$ in the free G-graded algebra $F\langle X_G\rangle$. The ungraded version of the theorem below was proved by Kemer (see [12], [13]).

Theorem 1.3 (Representability of G-graded affine algebras)**.** *Let W be a PI-affine algebra over a field F of characteristic zero and assume it is G-graded. Then there exists a field extension K of F and a finite-dimensional algebra A over K such that* $\mathrm{id}_G(W) = \mathrm{id}_G(A)$. *In particular* $\mathrm{id}(W) = \mathrm{id}(A)$.

Thus, in order to prove the conjecture for PI-affine G-graded algebras it is sufficient to prove it for an arbitrary finite-dimensional G-graded algebra A.

Remark 1.4. We may assume that the field F is algebraically closed. Indeed, if $\overline{F}$ is any extension of F (and in particular its algebraic closure) and if $\overline{A} = A \otimes_F \overline{F}$, then $\overline{A}$ is G-graded by $\overline{A}_g = A_g \otimes \overline{F}$ and one shows that the G-graded identities of $\overline{A}$ coincide with the G-graded identities of A (see [2, Remark 1.5],[9, Remark 1]). It follows that the n-th codimension $c_n(\overline{A})$ (resp. $c_n(\overline{A_e})$) over $\overline{F}$ coincides with the n-th codimension $c_n(A)$ (resp. $c_n(A)$) over F.

The problem can be reduced further, namely to the case where G is a simple group. We say that the conjecture holds for a group G if it holds for any algebra W which is G-graded.

Lemma 1.5. *Suppose that H is a normal subgroup of G. If the conjecture holds for H and for G/H, then it holds for G. Hence, it is sufficient to prove the conjecture for simple groups G.*

Proof. Assume that A is G-graded. Then it is G/H-graded in the obvious way. Note that the $\overline{e}$-component of A ($\overline{e} \in G/H$ is the projection of the identity element of G) coincides with $A_H = \oplus_{h\in H} A_h$ (with the G-grading). Hence we have

$$\exp(A) \leq |G/H|^2 \exp(A_H)$$

and

$$\exp(A_H) \leq |H|^2 \exp(A_e).$$

This gives

$$\exp(A) \leq |G|^2 \exp(A_e)$$

as desired. □

Remark 1.6. In fact, we will not use this reduction since our proof does not become "easier" if G is assumed to be a simple group. However, proving the conjecture for cyclic groups of prime order (and hence for solvable groups) is substantially simpler.

2. Proofs

In this section we prove Theorem 1.2. We assume (as we may), that A is a G-graded, finite-dimensional algebra over F and F is algebraically closed.

We need to consider two decompositions of A. One, as an ordinary algebra and the other as a G-graded algebra.

Let J denote the Jacobson radical of A and $\widetilde{A} \cong A/J$ the semisimple quotient. By the Wedderburn-Malcev principal theorem, there exists a semisimple subalgebra S of A, that is isomorphic to $\widetilde{A}$. So we can write $A \cong S \oplus J$ where the isomorphism is (only) as vector spaces. Furthermore, the subalgebra S may be decomposed into a direct product $S_1 \times \cdots \times S_d$ of simple algebras.

A similar decomposition holds if we consider A as a G-graded algebra. Indeed, it is well known that the Jacobson radical J of A is G-graded (see [6]). Furthermore there exists a semisimple subalgebra S_G in A that is G-graded and is supplementary to J as an F-vector space (see [15]). The algebra S_G may be decomposed into the direct product of G-simple algebras $(S_G)_1 \times \cdots \times (S_G)_q$. Clearly, we may decompose further each G-simple component into simple ones and get a decomposition as above. We see that every simple component S_i is a constituent of precisely one G-simple component $(S_G)_{j(i)}$. We refer to the G-simple algebra $(S_G)_{j(i)}$ as the G-graded envelope (or just the envelope) of S_i.

Let us recall now how $\exp(A)$ is computed in terms of the structure of A. Consider all possible *nonzero* products in A of the form $S_{i_1} J S_{i_2} \cdots J S_{i_r}$. For such a product consider the sum of the dimensions of the *different* simple algebras S_i that are represented in the product. Then it is known that $\exp(A)$ is equal to the maximal value of these dimensions (see [9], [10]). In particular this shows that $\exp(A)$ is an integer.

Applying linearity we can replace each S_i that appears in a nonzero product $S_{i_1} J S_{i_2} \cdots J S_{i_r}$ by the corresponding G-simple envelope and get a nonzero product $(S_G)_{j(i_1)} J (S_G)_{j(i_2)} \cdots J (S_G) j(i_r)$. Of course, there may be repetitions among the G-simple components that appear (even if there are no repetitions among the S_i's) but clearly the sum of the dimensions of the *different* G-simple components which appear in such nonzero product is at least the sum of the dimensions of the S_i's. Let $\exp^G_{conj}(A)$ be the maximal possible value obtained in this way (as sum of the dimensions of the different G-simple components that appear in a nonzero product).

Remark 2.1. We use the notation $\exp^G_{conj}(A)$ since it turns out to be equal to $\exp^G(A)$, the G-graded exponent of the G-graded algebra A, if G is abelian and *conjecturally* if G is arbitrary (see [1]). In fact, for our purposes, we will only need $\exp^G_{conj}(A)$ (and not $\exp^G(A)$). Nevertheless, for completeness, we recall the definition of $\exp^G(A)$. Let $\mathcal{W}_G = F\langle X, G\rangle$ be the free G-graded algebra on a countable set, $\mathrm{id}_G(A)$ the ideal of G-graded identities of A and $F\langle X, G\rangle / \mathrm{id}_G(A)$ the corresponding relatively free G-graded algebra. For $n = 1, 2, \ldots$, we denote by $c^G_n(A)$ the dimension of the subspace of $\mathcal{W}_G$ spanned by multilinear elements in n (G-graded) free generators. We call $\{c^G_n(A)\}_{n=1}^{\infty}$ the sequence of G-codimensions of A. The G-graded exponent of A, is given by the limit $\lim_{n\to\infty} \sqrt[n]{c^G_n(A)}$.

The following theorem is the main result of the paper.

Theorem 2.2. *With the above notation,* $\exp^G_{conj}(A) \leq |G|^2 \exp(A_e)$.

Theorem 1.2 follows from this since $\exp(A) \leq \exp^G_{conj}(A)$.

Our proof is based on a key result of Bahturin, Sehgal and Zaicev (see [3]) in which they explicitly describe the structure of G-graded, finite-dimensional G-simple algebras in terms of *fine* and *elementary* gradings.

Before stating their result recall that a G-grading on a matrix algebra $M_r(F)$ is said to be *elementary* if there exists an r-tuple $(g_{k_1}, \dots, g_{k_r}) \in G^{\times r}$ such that for any $1 \le i, j \le r$, the elementary matrix $e_{i,j}$ is homogeneous of degree $g_{k_i}^{-1} g_{k_j} \in G$. A G-grading on an algebra A over F is said to be *fine* if each homogeneous component A_g, is of dimension ≤ 1 (as an F-space).

Theorem 2.3 ([3]). *Let B be a G-simple algebra. Then there exists a subgroup H of G, a 2-cocycle $f : H \times H \longrightarrow F^*$ where the action of H on F is trivial, an integer r and an r-tuple $(g_{k_1}, \dots, g_{k_r}) \in G^{\times r}$ such that B is G-graded isomorphic to $C = F^f H \otimes M_r(F)$, where $C_g = \mathrm{span}_F\{u_h \otimes e_{i,j} : g = g_{k_i}^{-1} h g_{k_j}\}$. Here $F^f H = \sum_{h \in H} F u_g$ is the twisted group algebra of H over F with the 2-cocycle f and $e_{i,j} \in M_r(F)$ is the (i,j)-elementary matrix.*

In particular the idempotents $1 \otimes e_{i,i}$ as well as the identity of B are homogeneous of degree $e \in G$.

Remark 2.4. Note that the grading above induces G-gradings on the subalgebras $F^f H \otimes F$ and $F \otimes M_r(F)$ which are *fine* and *elementary* respectively.

For the clarity of the exposition it is convenient to "place" the e-component in the elementary grading of $M_r(F)$ as blocks along the diagonal. Indeed reordering the r-tuple we can assume the grading on $A \cong M_r$ is as follows:

Fix an ordering in the group G, $(g_1, \dots, g_n)$. Fix also a decomposition of r into the sum of n integers $r = r_1 + r_2 + \ldots + r_n$. Note that some r_i may be zero. Then we assign the value $g_1 = e$ to the first r_1 indices, the value g_2 to the next r_2 indices and so on. Note that this determines a G-grading on $M_r(F)$ where the e-blocks are concentrated along the diagonal. We get $|G|$ e-blocks (again, we allow 0×0-blocks) where the i-th block is of size r_i.

In the sequel, whenever we are given a G-simple algebra, we will assume the grading is given as in Theorem 2.3 where the elementary grading on $M_r(F)$ is as above.

Let us analyze the e-component of B (B is G-simple algebra with the G-grading as in Theorem 2.3). By the definition of the G-grading, a basis element $u_h \otimes e_{i,j}$ is in the e-component if and only if $e = g_{k_i}^{-1} h g_{k_j}$ or equivalently $g_{k_i} = h g_{k_j}$. This means that g_i and g_j represent the same *right* coset of H in G. Consider the family of right H-cosets in G. For convenience we order the elements of G (in the elementary grading of $M_r(F)$) according to these equivalence classes. As a consequence we see that the e-component of B consists of an algebra which is isomorphic to the direct product of matrix algebras $M_{t_1}(F) \times \cdots \times M_{t_l}(F)$ where $l = [G : H]$, t_i is the number of elements in the grading vector $(g_{k_1}, \dots, g_{k_r}) \in G^{\times r}$ that belong to the i-th (right) H-coset, for $i = 1, \dots, l$ (note that t_i may be zero). In particular every element of the form $1 \otimes e_{i,i}$ belongs to a unique simple component of B_e. We refer to this component as the e-block of B_e determined by the element $1 \otimes e_{i,i}$.

Let A be a G-graded finite-dimensional algebra. Consider the decomposition $A \cong S_G \oplus J$ where S_G is a semisimple algebra and J is the Jacobson radical. Let $S_G \cong (S_G)_1 \times \cdots \times (S_G)_q$ be the decomposition of S_G into the direct product of its G-simple components. In view of the above decomposition we will consider homogeneous elements that belong to J (radical elements) and basis elements of the form $u_h \otimes e_{i,j} \in F^f H \otimes M_r(F)$ (semisimple elements) where $F^f H \otimes M_r(F)$ is a G-simple component of S_G.

In the following lemma we construct suitable monomials in B which are nonzero products of basis elements of the form $u_h \otimes e_{i,j}$.

Lemma 2.5 (see proof of Lemma 5.2 in [2])**.** *Let $B = F^f H \otimes M_r(F)$ be a G-simple algebra (G-graded as above).*

(1) *There exists a nonzero G-graded monomial that consists of the product of all r^2 elementary matrices of $M_r(F)$ without repetitions. Moreover, for any $k = 1, \dots, r$, there is such a monomial which starts with the elementary matrix $e_{k,k}$ and ends by $e_{i,k}$ for some $i \neq k$. Consequently we have a nonzero monomial Z which consists of all r^2 elements of the form $1 \otimes e_{i,j}$ in B.*

(2) *For any $1 \leq k \leq r$ there exists a nonzero G-graded monomial $\widehat{Z}$ which consists of basis elements of $F^f H \otimes M_r(F)$ such that*
   (a) *It starts with $1 \otimes e_{k,k}$ and ends by an element of the form $u_h \otimes e_{i,k}$*
   (b) *For any $1 \leq i \leq r$ the idempotent $1 \otimes e_{i,i}$ appears in $\widehat{Z}$.*
   (c) *If $\widehat{Z}$ decomposes into the product $X \cdot 1 \otimes e_{i,i} \cdot Y$ where X has homogeneous degree $g \in G$, then for any $\widehat{g} \in g g_{k_i}^{-1} H g_{k_i}$ there exists a decomposition $\widehat{Z} = X_0 \cdot 1 \otimes e_{i,i} \cdot Y_0$ where X_0 has homogeneous degree $\widehat{g}$.*
   (d) *The total value of $\widehat{Z}$ in B is $1 \otimes e_{k,k}$.*

Proof. The first part is well known. In order to construct $\widehat{Z}$ we replace the idempotent $1 \otimes e_{i,i}$ in Z by the monomial

$$u_{h_1} \otimes e_{i,i} \cdot z \cdot u_{h_2} \otimes e_{i,i} \cdot z \cdots z \cdot u_{h_r} \otimes e_{i,i} \cdot z$$

where $z = 1 \otimes e_{i,i}$, $h_1 = 1$ and

$$(h_1, h_1 h_2, \dots, h_1 h_2 \cdots h_r)$$

consist of all elements of the group H. This takes care of parts (a) (b) and (c). The condition in part (d) is obtained by multiplying (on the right) by an element of the form $\lambda u_h \otimes e_{k,k}$.

□

Remark 2.6. Any semisimple element $u_h \otimes e_{i,j}$ may be multiplied from left (resp. right) by $1 \otimes e_{i,i}$ (resp. $1 \otimes e_{j,j}$) without changing its value and hence by the lemma, we can insert a G-graded monomial $\widehat{Z}$ (as in Lemma 2.5) from left (or right) and such that $\widehat{Z} \times u_h \otimes e_{i,j} = u_h \otimes e_{i,j}$ (or $u_h \otimes e_{i,j} \times \widehat{Z} = u_h \otimes e_{i,j}$). Furthermore since the element $1 \otimes e_{1,1}$ appears in the G-graded monomial $\widehat{Z}$ we can insert an additional G-graded monomial E of the same form which starts with $1 \otimes e_{1,1}$ and whose value is $1 \otimes e_{1,1}$. Thus, we can decompose the G-graded monomial $\widehat{Z}$ into a product XY such that

$$\widehat{Z} = XY = XEY$$

where $E \in B_e$ is as above. Moreover for every semisimple element $u_h \otimes e_{i,j}$ there exist G-graded monomials X, E, Y as above such that $XEY \times u_h \otimes e_{i,j} = u_h \otimes e_{i,j}$ (similarly we could find G-graded monomials which multiply $u_h \otimes e_{i,j}$ from the right).

Let

$$\Lambda = z_1 v_1 z_2 \cdots z_n v_n z_{n+1}$$

be a (nonzero) monomial in A that realizes the value of $\exp^G_{conj}(A)$. Here the z's are semisimple elements and the v's are radical. Note that we may assume that the semisimple elements belong to *different* G-simple components. Indeed, those that repeat may be "swallowed" by the radical elements.

Now, every semisimple element z_i may be multiplied (say from the right) by a G-graded monomial of semisimple elements (that belong to the same G-simple component) of the form $X_i E_i Y_i$ without changing the value of Λ. Furthermore, the G-graded monomial X_i may be "swallowed" by the radical elements v_i for $i = 2, \ldots, n+1$ whereas Y_j may be "swallowed" by the radical elements v_{j+1} for $j = 1, \ldots, n$. Finally, if we throw away X_1 and Y_{n+1} from the G-graded monomial we get a nonzero G-graded monomial

$$\Omega = E_1 v_1 E_2 \cdots E_n v_n E_{n+1}$$

with the following properties:

(1) For $i = 1, \ldots, n+1$, E_i is a G-graded monomial whose elements are in the i-th G-simple component (after renumbering). In particular $n + 1 \leq q$ where q is the total number of G-simple components of A.
(2) The (total) value of E_i is $1 \otimes e_{1,1}$ of the corresponding G-simple component. Furthermore, the G-graded monomial which corresponds to E_i starts with $1 \otimes e_{1,1}$.
(3) The G-graded monomial $\Omega = E_1 v_1 E_2 \cdots E_n v_n E_{n+1}$ realizes the value of $\exp^G_{conj}(A)$. We may refer to Ω as an element of A and also as a G-graded monomial whose elements are in A. We let $g_0 \in G$ be the homogeneous degree of Ω.

Definition 2.7. We say that a G-graded monomial $T = a_1 \cdots a_n$ of homogeneous elements ($a_i \in A$) has *no proper* g-submonomial if either $n = 1$ or else there is no word of the form $T^{'} = a_1 \cdots a_m$ for $m < n$ whose homogeneous degree is g.

For any g in G we consider a possible decomposition of

$$\Omega = X_g \Sigma_1 \Sigma_2 \cdots \Sigma_d Y_{g^{-1}g_0}$$

where

(1) X_g has homogeneous degree g and has no proper g-submonomial.
(2) For $i = 1, \ldots, d$, Σ_i has homogeneous degree e and has no proper e-submonomial.
(3) $Y_{g^{-1}g_0}$ has homogeneous degree $g^{-1}g_0$ and has no proper e-submonomial.

Remark 2.8. (1) The decomposition above is maximal and unique in the sense that there is no other decomposition with the same structure and the same or more number of words (i.e. starting with a word of homogeneous degree g, followed by words of homogeneous degree e and ending by a word of homogeneous degree $g^{-1}g_0$).
(2) Note that such a decomposition may not exist (indeed X_g may not exist). Note also that d may be zero, in which case we have $\Omega = X_g Y_{g^{-1}g_0}$. In what follows there is no need to consider these cases separately (including the case where the decomposition does not exist).

Given such a decomposition (for $g \in G$) we consider the semisimple elements (idempotents) of the form $1 \otimes e_{i,i}$ which can be inserted in between any two adjacent words and giving nonzero product. Note that these idempotents are uniquely determined. We call these idempotents " e-stops".

Remark 2.9. (1) Note that the different e-stops may or may not belong to the same G-simple component.

(2) As mentioned above if an e-stop belongs to a G-simple component B then it determines uniquely an e-block of B_e.
(3) e-stops that belong to the same G-simple component determine the same e-block of B_e.
(4) A G-simple component may not be represented by an e-stop.
(5) In case X_g does not exists, the set of e-stops is empty.

Now for any $g \in G$, note that the G-graded monomial

$$\Omega = X_g \Sigma_1 \Sigma_2 \cdots \Sigma_d Y_{g^{-1}g_0}$$

gives rise to the *nonzero* G-graded monomial

$$(1 \otimes e_{i_1,i_1})_{s_1} \Sigma_1 (1 \otimes e_{i_2,i_2})_{s_2} \Sigma_2 \cdots \Sigma_d (1 \otimes e_{i_{d+1},i_{d+1}})_{s_{d+1}}.$$

Here the idempotent $(1 \otimes e_{i_j,i_j})_{s_j}$, $j = 1, \ldots, d+1$, belongs to $(S_G)_{s_j}$, (the s_j-th G-simple component in the decomposition of A). The reader should keep in mind that all indices depend on $g \in G$.

Let us consider the above monomial as a monomial in A_e. Clearly, since the monomial is nonzero, it provides a lower bound to $\exp(A_e)$. Let us summarize the above considerations:

(1) The value of $\exp^G_{Conj}(A)$ is realized by the (nonzero) G-graded monomial
$$\Omega = E_1 v_1 E_2 \cdots E_n v_n E_{n+1}.$$
(2) For any g in G we consider the decomposition
$$\Omega = X_g \Sigma_1 \Sigma_2 \cdots \Sigma_d Y_{g^{-1}g_0}$$
and the corresponding nonzero monomial in A_e
$$(1 \otimes e_{i_1,i_1})_{s_1} \Sigma_1 (1 \otimes e_{i_2,i_2})_{s_2} \Sigma_2 \cdots \Sigma_d (1 \otimes e_{i_{d+1},i_{d+1}})_{s_{d+1}}.$$

As mentioned above each e-stop $(1 \otimes e_{i_j,i_j})_{s_j}$ determines uniquely an e-block of the G-simple component $(S_G)_{s_j}$. Note also (by the construction of the word Ω) that e-stops that belong to the same G-simple component are adjacent to each other. In other words once we "leave" a G-simple component $(S_G)_m$ we do not return to it. Consequently, all e-stops that belong to the same G-simple component determine the same e-block.

As a result of this we have that each configurations (arising from different elements in G) provides a lower bound to $\exp(A_e)$. More precisely:

Corollary 2.10. *Denote by $(B_{g,1}, \ldots, B_{g,n+1})$ the e-blocks (in the different G-simple components) determined by $g \in G$ and let $b^2_{g,i} = \dim_F(B_{g,i})$. Then*

$$\exp(A_e) \ \geq b^2_{g,1} + \cdots + b^2_{g,n+1}.$$

Suppose now we start with a different element $\widehat{g}$ of G. Consider the words $X_{\widehat{g}}$, $Y_{\widehat{g}^{-1}g_0}$ and Σ's. These determine e-stops and consequently e-blocks. In the next lemma we establish the relation between elements of G which determine the same e-block in a G-simple component. Let us denote by H_m the group H which appears in the G-graded structure of the G-simple component $(S_G)_m$ (see Theorem 2.3).

Lemma 2.11. *The number of elements in G which determine the same (nonzero) e-block in $(S_G)_m$ is precisely $ord(H_m)$.*

Proof. First note that each (nonzero) e-block in $(S_G)_m$ is determined by some $g \in G$. Suppose now that g and $\widehat{g}$ determine e-stops $1 \otimes e_{i,i}$ and $1 \otimes e_{j,j}$ in $(S_G)_m$ respectively. This means that $\widehat{g} = gg_{k_i}^{-1}hg_{k_j}$ for some $h \in H_m$. But the idempotents $1 \otimes e_{i,i}$ and $1 \otimes e_{j,j}$ determine the same e-block in $(S_G)_m$ (and hence g and $\widehat{g}$ determine the same e-block in $(S_G)_m$) if and only if $H_m g_{k_i} = H_m g_{k_j}$. Replacing g_{k_j} by an element of the form $h^{'}g_{k_i}$, $h^{'} \in H_m$, we see that g and $\widehat{g}$ determine the same e-block in $(S_G)_m$ if and only if they represent the same left coset of $g_{k_i}^{-1}H_m g_{k_i}$ in G. In order to complete the proof of the lemma it remains to show that if g determines an e-stop in $(S_G)_m$ then all elements in $gg_{k_i}^{-1}H_m g_{k_i}$ determine e-stops in $(S_G)_m$ as well. But this follows from Lemma 2.5 and so the lemma is proved. $\square$

The main point of the lemma is the following

Corollary 2.12. *Consider the different decompositions of Ω*

$$\Omega = X_g \Sigma_1 \Sigma_2 \cdots \Sigma_d Y_{g^{-1}g_0}$$

where g runs over all elements of G. Then each (nonzero) e-block in any G-simple component $(S_G)_m$ will be represented by the corresponding e-stops precisely $ord(H_m)$ times. In the calculation below we include the e-blocks of dimension 0.

Let us show now that $\exp^G_{conj}(A)$ is bounded (from above) by $|G|^2 \exp(A_e)$. Suppose not. Then

$$\exp^G_{conj}(A) > |G|^2 \exp(A_e)$$

and hence, for every $g \in G$ we have

$$\exp^G_{conj}(A) > |G|^2 (b^2_{g,1} + \cdots + b^2_{g,n+1}).$$

Summing over all elements of G, we obtain

$$|G| \exp^G_{conj}(A) > \sum_{g \in G} |G|^2 (b^2_{g,1} + \cdots + b^2_{g,n+1}).$$

Recall now that every G-simple algebra $(S_G)_i$ has $r_i = [G : H_i]$ e-blocks (including the 0×0-blocks). In order to simplify the notation we rename the e-blocks as follows: the e-blocks of the i-th G-simple component $(S_G)_i$ will be denoted by $B_{i,1}, \ldots, B_{i,r_i}$ and $b^2_{i,j} = \dim_F(B_{i,j})$.

Now the left hand side yields

$$|G| \exp^G_{conj}(A) = |G| \times \sum_{i=1}^{n+1} |H_i| (b_{i,1} + \cdots + b_{i,r_i})^2$$

whereas the right hand side yields

$$|G|^2 \sum_{i=1}^{n+1} |H_i|(b_{i,1}^2 + \cdots + b_{i,r_i}^2).$$

In order to complete the proof we need to show that the inequality

$$|G| \times \sum_{i=1}^{n+1} |H_i|(b_{i,1} + \cdots + b_{i,r_i})^2 > |G|^2 \sum_{i=1}^{n+1} |H_i|(b_{i,1}^2 + \cdots + b_{i,r_i}^2)$$

is not possible. Clearly, this will follow if for every i we have

$$|G| \times |H_i|(b_{i,1} + \cdots + b_{i,r_i})^2 \leq |G|^2 |H_i|(b_{i,1}^2 + \cdots + b_{i,r_i}^2).$$

We simplify the notation once again by putting $H = H_i$, $r = r_i$ and $b_j = b_{i,j}$ for $j = 1, \ldots, r$. We therefore need to show that the inequality

$$(b_1 + \cdots + b_r)^2 \leq |G|(b_1^2 + \cdots + b_r^2)$$

holds. Clearly, it suffices to show that

$$(b_1 + \cdots + b_r)^2 \leq [G : H](b_1^2 + \cdots + b_r^2) = r(b_1^2 + \cdots + b_r^2)$$

and this is clear since it is equivalent to

$$\sum_{i<j} (b_i - b_j)^2 \geq 0.$$

This completes the proof of Theorem 2.2 and hence of Theorem 1.2.

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Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail address: aljadeff@tx.technion.ac.il