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GROWTH AND UNIQUENESS OF RANK

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ABSTRACT

We prove that algebras of sub-exponential growth and, more generally, rings with a sub-exponential "growth structure" have the unique rank property. In the opposite direction the proof shows that if the rank is not unique one gets lower bounds on the exponent of growth. Fixing the growth exponent it shows that an isomorphism between free modules of greatly differing ranks can only be implemented by matrices with entries of logarithmically proportional high degrees.

It is well known that there exist rings over which the rank of a free left module is not uniquely defined, i.e., they have modules that are free on two bases of different cardinalities. A ring for which this distressing phenomenon does not happen is said to have the (left) *unique rank property*; we also say that it has the "UR" property or, simply, that it "has UR". A commutative ring always has the UR property since it has a non-trivial (i.e., with 1 going to 1) homomorphism into a field. More generally a ring has the UR property whenever it has a non-trivial homomorphism into a ring that has the UR property. But many rings to which our theorem (below) applies do not have (apriori) such homomorphisms.

It is easy to see that if a ring R has a subring S which has UR and R is a finitely generated free module over S then R has UR. In this note we generalize this fact in a somewhat unexpected direction. We show that if S satisfies some restrictions (for example S may be Noetherian) and R is generated, over S, in a certain "controlled" way then R still has the UR property. The most general condition on S is the content of the next definition. It generalizes commutativity (which was our assumption in an earlier draft).

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