Intensity control with a free-form lens

Jacob Rubinstein and Gershon Wolansky
Mathematics Department, Indiana University, Bloomington, IN 47405, USA, and Department of Mathematics, Technion, Haifa 32000, Israel

Abstract.
A family of free-form lenses for intensity control is designed. The lens can shape an incident collimated beam with a given intensity distribution I_1 into a new collimated beam with intensity distribution I_2. No symmetry is assumed for the two intensity profiles. The key idea is that the lens design problem can be formulated and solved in terms of an optimization process involving a specific action functional. It is further shown that the free-form lens can be manufactured by a surfacing process using a convex tool.

OCIS 080.2470, 140.3300

1 Introduction

The problem of intensity control in general and beam shaping in particular is important in laser applications and in other fields. The activity in this area has grown considerably in the last two decades. The basic theory and some of the applications are summarized for example in the review paper [1] and in the book [2]. Almost all the known designs are based on symmetric optical elements. The goal of this paper is to introduce a new type of lenses that can shape a beam with an arbitrary intensity profile.

In an illumination design the designer needs to control the phase (or the rays) and the propagation of the intensity along the rays. The problems of ray and intensity propagation are closely connected: The intensity transport equation can be integrated into a formula that relates the intensity at two points along a ray path in terms of the Jacobian of the ray mapping (see [3] and also equation (4) below). An equivalent integration formula expresses the intensity transport in terms of the Gaussian curvature of the wavefront [3]. Because of the mathematical nature of the transport equation, intensity control is often analyzed by differential geometry tools, such as the prescribed Gaussian curvature problem or the Monge-Ampere equation. This was first demonstrated by Keller in his pioneering study of reflectors [4].

Since illumination design is hard, it is typically solved under symmetry assumptions. The problem is thus reduced to just one spatial dimension which enables the formulation of appropriate, relatively easy to solve, ordinary differential equations. An exceptional case that is relevant to our design here is the two-reflector system described in [5]. Glimm and Oliker prove there that one can convert an incident collimated beam with intensity profile I_1 into another collimated beam with intensity profile I_2 with two reflectors. The authors showed that instead of solving a Monge-Ampere partial differential equation, one can use an optimization approach; thus the reflector surfaces are found by solving a minimization problem.

The goal of our paper is to show that a similar design can be achieved with a lens, that is, a homogeneous optical medium confined between two surfaces immersed in another homogeneous medium such as air. In the next section we formulate the differential equations for the ray mapping that ensure that the incident beam and the refracted beam are both collimated. In Section 3 we derive a variational formulation of the full intensity control problem. A two-reflectors design involves the quadratic Monge mass transport functional [5]. We show here that beam shaping lens design can be expressed in terms of an optimization process. The functional that needs to be optimized here is entirely different from the reflectors case. Amazingly, it turns out to be closely related to the weighted least action functional that is associated with the Klein-Gordon equation [6]. In Section 4 we discuss our results and describe an algorithm for computing the new lens. Finally we consider the fabrication issue and show that the new lens can be manufactured with a convex tool. However, unlike the two-reflector case of [5], the surfaces themselves need not be convex.

2 Formulation of the illumination problem

Consider a wave propagating towards the positive z direction. The wavefront at the plane $z = 0$ is assumed to be planar and orthogonal to the z axis, i.e. the phase is constant at $z = 0$. The plane orthogonal to the z axis is parameterized by $\mathbf{x} = (x, y)$. The wave's intensity at the plane $z = 0$ is prescribed by $I_1(\mathbf{x})$. The wave is refracted at the two surfaces f and g of a lens with refractive index n. For simplicity it is assumed that the lens is immersed in air where the refractive index is 1 and that $n > 1$. The goal is to design a lens such that after the refraction the emerging wave will have again a planar wavefront, and will have a prescribed intensity profile $I_2(\mathbf{x})$ at the plane $z = h$. The reflected intensity and energy absorption by the lens are neglected. Conservation of radiation implies the constraint

$$\int\int I_1(\mathbf{x})d\mathbf{x} = \int\int I_2(\mathbf{x})d\mathbf{x}. \tag{1}$$

We can now formulate the design problem:

Design Goal: Design a lens, consisting of a pair of surfaces $f(\mathbf{x})$ and $g(\mathbf{x})$, such that an incident beam with a planar wavefront orthogonal to the z axis and with an arbitrary intensity profile $I_1(\mathbf{x})$ is converted by the lens into an outgoing beam, also having a planar wavefront orthogonal to the z axis and with an arbitrary intensity profile $I_2(\mathbf{x})$.

The key tool for the design of this lens is the ray mapping

$$\mathbf{x}^{'} = T(\mathbf{x}) = (x^{'}(x,y), y^{'}(x,y)) \tag{2}$$

which gives the terminal position $\mathbf{x}^{'}$ on the plane $z = h$ of a ray that starts at a point $\mathbf{x}$ on the plane $z = 0$. A related geometric object is the ray mapping deviation

$$(\delta(\mathbf{x}), \varepsilon(\mathbf{x})) = \mathbf{x}^{'} - \mathbf{x} = (x^{'} - x, y^{'} - y). \tag{3}$$

Energy conservation implies that the ray mapping T must satisfy the condition [3]

$$I_2(T(\mathbf{x}))|J(T(\mathbf{x}))| = I_1(\mathbf{x}), \tag{4}$$

where J is the Jacobian of the ray mapping. A mapping T that satisfies (4) is said to *transport* I_1 *into* I_2. We denote this property by $T_{\#}I_1 = I_2$.

A mapping T transporting I_1 into I_2 does not have to be differentiable or even continuous. A generalization of (4) is:

Definition 2.1. *A mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ is said to transport an integrable function I_1 on $\mathbb{R}^2$ into another integrable function I_2 on $\mathbb{R}^2$ if, for any closed set $A \subset \mathbb{R}^2$,*

$$\int\int_{T^{-1}(A)} I_1(\mathbf{x})d\mathbf{x} = \int\int_A I_2(\mathbf{x})d\mathbf{x}.$$

Equivalently, for any continuous function ϕ of compact support in $\mathbb{R}^2$:

$$\int\int \phi(T(\mathbf{x}))I_1(\mathbf{x})d\mathbf{x} = \int\int \phi(\mathbf{x})I_2(\mathbf{x})d\mathbf{x}. \tag{5}$$

If T is a ray mapping that transports I_1 to I_2, then the surfaces f and g can be found by solving a set of differential equations that were derived by us in [7]:

$$\frac{\partial f}{\partial x} = \frac{n\delta}{R - n(g' - f)}, \quad \frac{\partial f}{\partial y} = \frac{n\varepsilon}{R - n(g' - f)} \tag{6}$$

$$\frac{\partial g'}{\partial x} = n\frac{\delta \partial x^{'}/\partial x + \varepsilon \partial y^{'}/\partial x}{R - n(g' - f)}, \quad \frac{\partial g'}{\partial y} = n\frac{\delta \partial x^{'}/\partial y + \varepsilon \partial y^{'}/\partial y}{R - n(g' - f)}. \tag{7}$$

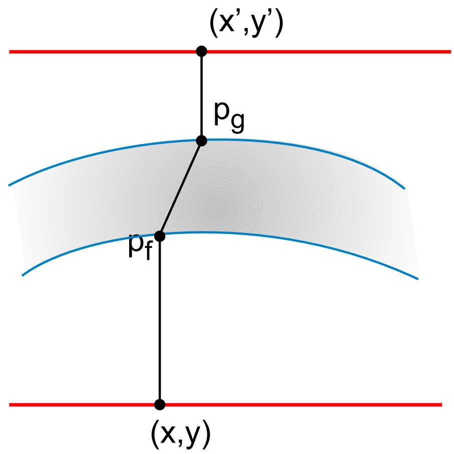

Figure 1: A sketch of the refraction of an individual ray in the beam

Here $g' = g(\mathbf{x}')$, $g' - f$ is the z component of the ray vector that connects the points $p_g = (x', y', g(x', y'))$ and $p_f = (x, y, f(x, y))$, while R is the distance between these two points. The geometrical picture is sketched in Figure 1.

While there are infinitely many ray mappings that satisfy (4), it is known [7] that a system of the form (6)-(7) is solvable if certain compatibility conditions hold. We shall now show that these compatibility conditions give rise to a specific constraint on the ray mapping T.

The first step is to related R to $g' - f$. For this purpose observe that the optical path length l of a ray starting at $(x, y, 0)$ and ending at (x', y', h) is

$$l = f + nR + h - g'. \tag{8}$$

Therefore $R = (g' - f)/n + (l - h)/n$. Since both $z = 0$ and $z = h$ are wavefronts, the optical path length l is constant. We insert identity (8) into (6)-(7), subtract the first pair of equations from the second pair, and define the constants $b = n - 1/n$ and $c = (l - h)/n$. We obtain

$$\frac{\partial(g' - f)}{\partial x} = n\frac{\delta\partial\delta/\partial x + \varepsilon\partial\varepsilon/\partial x}{c - b(g' - f)}, \quad \frac{\partial(g' - f)}{\partial y} = n\frac{\delta\partial\delta/\partial y + \varepsilon\partial\varepsilon/\partial y}{c - b(g' - f)}. \tag{9}$$

After a little algebra the system (9) can be expressed as

$$\begin{aligned} \frac{\partial}{\partial x}\left(-b(g' - f)^2 + 2c(g' - f)\right) &= n\frac{\partial}{\partial x}\left(\delta^2 + \varepsilon^2\right), \\ \frac{\partial}{\partial y}\left(-b(g' - f)^2 + 2c(g' - f)\right) &= n\frac{\partial}{\partial y}\left(\delta^2 + \varepsilon^2\right). \end{aligned} \tag{10}$$

Therefore $g' - f$ is related to $\delta^2 + \varepsilon^2$ through the quadratic equation

$$-b(g' - f)^2 + 2c(g' - f) - n(\delta^2 + \varepsilon^2) + a = 0, \tag{11}$$

where a is a constant. Solving for $(g' - f)$ we write

$$g(\mathbf{x}'(\mathbf{x})) - f(\mathbf{x}) = \frac{c}{b} \pm \frac{1}{b}\left(\chi - bn(\delta^2(\mathbf{x}) + \varepsilon^2(\mathbf{x}))\right)^{1/2}, \tag{12}$$

where $\chi = c^2 + ab$. Substitution of this solution into (6) gives

$$\frac{\partial f}{\partial x} = \mp \frac{n\delta}{\left(\chi - bn(\delta^2 + \varepsilon^2)\right)^{1/2}}, \quad \frac{\partial f}{\partial y} = \mp \frac{n\varepsilon}{\left(\chi - bn(\delta^2 + \varepsilon^2)\right)^{1/2}}. \tag{13}$$

Snell's law implies that when $n > 1$ then ∇f and $T(\mathbf{x}) - \mathbf{x}$ point in opposite directions. Therefore we must choose the minus sign in (13) and obtain finally

$$\frac{\partial f}{\partial x} = \frac{-n\delta}{\left(\chi - bn(\delta^2 + \varepsilon^2)\right)^{1/2}}, \quad \frac{\partial f}{\partial y} = \frac{-n\varepsilon}{\left(\chi - bn(\delta^2 + \varepsilon^2)\right)^{1/2}}. \tag{14}$$

Equation (14) indicates that the ray mapping T indeed cannot be arbitrary. Rather, the deviation vector $(\delta, \varepsilon) = T(\mathbf{x}) - \mathbf{x}$ must be such that the right hand side of the system (14) is a gradient. The system (13) was also derived by a slightly different method by Davidson et al. [8] in their work on optical coordinate transformations.

3 A variational characterization of the ray mapping

The main problem is to find a ray mapping T that satisfies *both* the energy conservation constraint (4) and the ray refraction equation (14). We now show that this task can be achieved by minimizing an appropriate cost function. In addition to characterizing in this way the existence of a feasible ray mapping, the optimization problem can be used to actually compute T, and then, through the surface equations (6)-(7) to construct the desired lens.

The optimization problem that we derive is related to the weighted least action (WLA) concept that was introduced in [10]. The specific action that we need here is the one associated with the Klein-Gordon equation. Define the cost function (Lagrangian)

$$D^*(\mathbf{s}) = \begin{cases} -\frac{c}{b} - \frac{1}{b}\left(\chi - bn|\mathbf{s}|^2\right)^{1/2} & \text{if } |\mathbf{s}| < \sqrt{\chi/bn} \\ \infty & \text{if } |\mathbf{s}| \geq \sqrt{\chi/bn}. \end{cases} \tag{15}$$

Notice that the definition above means that D^* is a convex function. For any ray mapping $\mathbf{x}' = T(\mathbf{x})$ we define the associated WLA

$$\mathbf{M}(T) := \int\int I_1(\mathbf{x}) D^*(|\mathbf{x} - T(\mathbf{x})|) d\mathbf{x} \ . \tag{16}$$

It will be convenient to look also at a related definition:

Definition 3.1. *A density function* $\lambda(\mathbf{x}, \mathbf{y})$ *on* $\mathbb{R}^2 \times \mathbb{R}^2$ *is said to be in the space* $\Lambda(I_1, I_2)$ *if its* $\mathbf{x}$ *and* $\mathbf{y}$ *marginal densities are given by* I_1 *and* I_2*, respectively. That is*

$$\int\int\int\int (\phi(\mathbf{x}) + \psi(\mathbf{y}))\, \lambda(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \int\int I_1(\mathbf{x})\phi(\mathbf{x}) d\mathbf{x} + \int\int I_2(\mathbf{x})\psi(\mathbf{x}) d\mathbf{x} \tag{17}$$

for any pair of continuous functions ϕ, ψ *with compact support in* $\mathbb{R}^2$.

For any density function λ in Λ we define the *Kantorovich Relaxation* of the WLA to be

$$\mathbf{K}(\lambda) := \int\int\int\int D^* \left(\left| \mathbf{x} - \mathbf{x}^{'} \right| \right) \lambda(\mathbf{x}, \mathbf{x}^{'})\; d\mathbf{x} d\mathbf{x}^{'}, \tag{18}$$

We now argue

Theorem 1. *Any mapping* T *that is a critical point of the action* $\mathbf{M}$ *under the constraint* $T_{\#} I_1 = I_2$ *achieves the Design Goal in the sense that the surfaces* $f(\mathbf{x})$ *and* $g(\mathbf{x})$ *are determined by*

$$\frac{-n\,(T(\mathbf{x}) - \mathbf{x})}{\left(\chi - bn\,|T(\mathbf{x}) - \mathbf{x}|^2\right)^{1/2}} = \nabla f(\mathbf{x}), \quad \frac{-n\,(\mathbf{x} - T^{-1}(\mathbf{x}))}{\left(\chi - bn\,|T^{-1}(\mathbf{x}) - \mathbf{x}|^2\right)^{1/2}} = \nabla g(\mathbf{x}) \tag{19}$$

Proof: We need to compute the first variation of $\mathbf{M}$ under the constraint (4). The computation of the first variation is based on a useful decomposition of intensity transporting mappings T [9]. Let T be a fixed mapping satisfying (4), and let $\mathcal{S}$ be the set of mappings from the plane to itself that transports I_1 to itself, i.e. $S_{\#} I_1 = I_1$ for all $S \in \mathcal{S}$. Then any mapping U transporting I_1 to I_2 can be written as $U = T \circ S^{-1}$ for some $S \in \mathcal{S}$. To facilitates the computation of the first variation we replace $\mathbf{M}$ by the equivalent functional (18) with the special choice

$$\lambda_U(\mathbf{x}, \mathbf{x}^{'}) = I_1(\mathbf{x}) \delta(\mathbf{x}^{'} - U(\mathbf{x}))\ . \tag{20}$$

Since $U_{\#} I_1 = I_2$ then $\lambda_U \in \Lambda(I_1, I_2)$. We shall slightly abuse the notation and write $\mathbf{K}(U)$ instead of $\mathbf{K}(\lambda_U)$. Since T is fixed, it follows that $\mathbf{K}(U)$ depends, in fact, on $S \in \mathcal{S}$. Setting $U_{(S)} := T \circ S^{-1}$, we can write

$$\begin{aligned} \mathbf{K}\left(U_{(S)}\right) &= \int\int\int\int D^* \left(\left| \mathbf{x} - \mathbf{x}^{'} \right| \right) I_1(\mathbf{x}) \delta(\mathbf{x}^{'} - T\left(S^{-1}(\mathbf{x})\right) d\mathbf{x} d\mathbf{x}^{'} \\ &= \int\int D^* \left(\left| \mathbf{x} - T\left(S^{-1}(\mathbf{x})\right) \right| \right) I_1(\mathbf{x}) d\mathbf{x}\ . \end{aligned} \tag{21}$$

The computation of the first variation of $\mathbf{K}$ requires exploring a neighborhood of the mapping U. It is convenient to introduce for this purpose the parameter τ and use it to parameterize the family of mapping S; namely, we consider the orbit in $\mathcal{S}$ given by $S^{(\tau)}(\mathbf{x})$ for $\tau \in \mathbb{R}$ so that $S^{(0)}$ is the identity mapping. The 'velocity' of the flow $S^{(\tau)}(\mathbf{x})$ is expressed in the form

$$\frac{d}{d\tau}S^{(\tau)}(\mathbf{x}) = \mathbf{w}\left(S^{(\tau)}(\mathbf{x})\right) \ . \tag{22}$$

The condition $S^{(\tau)} \in \mathcal{S}$ for all $\tau \in \mathbb{R}$ implies that $\mathbf{w}$ satisfies the equation

$$\nabla \cdot (I_1(\mathbf{x})\mathbf{w}(\mathbf{x})) = 0. \tag{23}$$

We now substitute $S^{(\tau)}$ into (21). Since $S^{(\tau)}_{\#} I_1 = I_1$ we can use the transport relation (5) (with $S^{(\tau)}$ instead of T, and I_1 as the target intensity instead of I_2). We obtain

$$\begin{aligned}\mathbf{K}\left(U_{(S^{(\tau)})}\right) &= \int\int D^*\left(\left|\mathbf{x} - T\left(\left[S^{(\tau)}\right]^{-1}(\mathbf{x})\right)\right|\right) I_1(\mathbf{x})d\mathbf{x} \\ &= \int\int D^*\left(\left|S^{(\tau)}(\mathbf{x}) - T(\mathbf{x})\right|\right) I_1(\mathbf{x})d\mathbf{x}\end{aligned} \tag{24}$$

Differentiating (24) at the point $\tau = 0$ and using (22) gives

$$\begin{aligned}\frac{d}{d\tau}\mathbf{K}\left(U_{(S^{(\tau)})}\right)_{\tau=0} &= \int\int \nabla D^*\left(|\mathbf{x} - T(\mathbf{x})|\right) \cdot \mathbf{w}(\mathbf{x}) I_1(\mathbf{x})d\mathbf{x} \\ &= \int\int \left(D^*\right)'\left(|\mathbf{x} - T(\mathbf{x})|\right) \frac{\mathbf{x} - T(\mathbf{x})}{|\mathbf{x} - T(\mathbf{x})|} \cdot \mathbf{w}(\mathbf{x}) I_1(\mathbf{x})d\mathbf{x} \ .\end{aligned} \tag{25}$$

In particular, $T = U_{S^{(0)}}$ is a critical point of $\mathbf{M}$ if and only if the right hand side of (25) is zero for *any* $\mathbf{w}$ satisfying (23). The Helmholtz decomposition of vector fields implies, then, that the term multiplying $I_1\mathbf{w}$ in (25) must be a gradient, i.e

$$\left(D^*\right)'\left(|\mathbf{x} - T(\mathbf{x})|\right) \frac{\mathbf{x} - T(\mathbf{x})}{|\mathbf{x} - T(\mathbf{x})|} = \nabla\Phi(\mathbf{x}) \tag{26}$$

for some potential Φ. Using (15) we obtain that (26) is compatible with (14) upon setting $\Phi = f$. This also yields the equation of ∇f in (19). The equation for ∇g in (19) follows from a symmetry argument that we invoke in the next section.

4 Implementation and Discussion

We have shown that a free-form lens can shape an arbitrary collimated beam. In this section we discuss the result and in particular indicate how to compute and manufacture such a lens.

Any critical point of the functional $\mathbf{M}$ provides a beam shaping lens according to our initial goal. However, it is difficult to characterize such functions in general. Fortunately, the case of the global minimizer for the WLA functional $\mathbf{M}$ is different. Therefore we shall concentrate now on this case.

We first consider the numerical minimization of $\mathbf{M}$. A discussion on algorithms for weighted least action functionals was given in [10]. In particular we highlight the steepest decent algorithm proposed in [11]. Since the derivation is similar to the action of the Fresnel equation that was studied in [12], we just give the result without spelling out the details. Consider, therefore, an initial ray mapping $U(\mathbf{x},0)$ that transports I_1 to I_2. The function $U(\mathbf{x},0)$ serves as initial data for a flow

$$I_1\frac{\partial}{\partial t}U(\mathbf{x},t)+V(\mathbf{x},t)\nabla U(\mathbf{x},t)=0, \tag{27}$$

where $V(\mathbf{x},t)$ is the divergence-free part in the Helmholtz decomposition of $U(\mathbf{x},t)$:

$$\frac{n\left(U(\mathbf{x},t)-\mathbf{x}\right)}{\left(\chi+bn|U(\mathbf{x},t)-\mathbf{x}|^2\right)^{1/2}}=\nabla P(\mathbf{x},t)+V(\mathbf{x},t),\quad \nabla\cdot V(\mathbf{x},t)=0. \tag{28}$$

The cost function $\mathbf{M}$ decreases along the flow of mappings $U(\mathbf{x},t)$, and in addition the entire flow $U(\mathbf{x},t)$ transports I_1 into I_2. Therefore the flow is expected to converge to the ray mapping $\bar{T}$ that is the global minimizer of $\mathbf{M}$. A code that implements a pair of evolution equations of the type (27)-(28), where the denominator in the left hand side of (28) is replaced by 1 is described in [12].

We turn our attention to the manufacturing question. While free-form lenses can be useful in achieving difficult design goals, one must make sure that they can be fabricated by available tools. It is desired in general to have lenses where both surfaces are convex (or concave). It is not guaranteed that a beam shaping lens like the one we derive here would always have convex or concave surfaces. However, we can show that both lens surfaces f and g can be fabricated with a convex tool.

To justify the last statement let us recall an important property of weighted least action functionals that was derived in some generality in [6] (see also [13]). The Lagrangian D^* is associated with a Hamiltonian D, which in the case of the Klein-Gordon equation is given by

$$D(\nabla\phi)=\max_{\mathbf{s}}\left(\mathbf{s}\cdot\nabla\phi-D^*(\mathbf{s})\right)=\frac{c}{b}+\frac{\chi^{1/2}}{b}\left(1+b|\nabla\phi|^2\right)^{1/2}. \tag{29}$$

Consider further the Hamilton-Jacobi equation

$$\frac{\partial\phi}{\partial t}+D(\nabla\phi)=0, \tag{30}$$

in conjunction with the transport equation

$$\frac{\partial I}{\partial t}+\nabla(v(\mathbf{x},t)I)=0,\quad v(\mathbf{x},t)=D'(|\nabla\phi|)\frac{\nabla\phi}{|\nabla\phi|}, \tag{31}$$

where D' denotes the derivative of D with respect to its argument.

The solution of equation (30) is given by the Lax-Hopf formula [14]

$$\phi(\mathbf{x}, t) = \min_{\mathbf{y}} (D^*(|\mathbf{x} - \mathbf{y}|) + \phi(\mathbf{y}, 0)). \tag{32}$$

Consider now a pair $\phi_1,\ \phi_2$ satisfying

$$\phi_2(\mathbf{y}) - \phi_1(\mathbf{x}) \leq D^*(\mathbf{x} - \mathbf{y}). \tag{33}$$

Then, for any mapping T that transports I_1 to I_2

$$M(T) \geq \int\int (I_2(\mathbf{x})\phi_2(\mathbf{x}) - I_1(\mathbf{x})\phi_1(\mathbf{x}))\, d\mathbf{x}. \tag{34}$$

Indeed, thanks to (5)

$$\int\int (I_2\phi_2 - I_1\phi_1)\, d\mathbf{x} = \int\int I_1(\mathbf{x})\, (\phi_2(T(\mathbf{x})) - \phi_1(\mathbf{x}))\, d\mathbf{x} \leq \int\int I_1(\mathbf{x}) D^*(|T(\mathbf{x}) - \mathbf{x}|) d\mathbf{x} = M(T). \tag{35}$$

It was shown in [13] and in [10] that if the pair ϕ_1, ϕ_2 maximizes the right hand side of (34) subjected to the constraint (33), then the solution (32) to equation (30) with $\phi(\mathbf{x}, 0) = \phi_1(\mathbf{x})$ satisfies $\phi(\mathbf{x}, 1) = \phi_2(\mathbf{x})$. Moreover, substituting this solution into (31) with initial conditions $I(\mathbf{x}, 0) = I_1(\mathbf{x})$ implies $I(\mathbf{x}, 1) = I_2(\mathbf{x})$. In addition, the optimal pair ϕ_1, ϕ_2 satisfies

$$\phi_2(\mathbf{y}) - \phi_1(\mathbf{x}) = D^*(\mathbf{x} - \mathbf{y}). \tag{36}$$

Recalling (12) and (15), and identifying $\phi_1 = -f$, and $\phi_2 = -g$, we obtain

$$g(\mathbf{x}) = \max_{\mathbf{y}} \left(f(\mathbf{y}) - D^*(|\mathbf{x} - \mathbf{y}|) \right), \tag{37}$$

and similarly

$$f(\mathbf{x}) = \min_{\mathbf{y}} \left(D^*(\mathbf{x} - \mathbf{y}) - g(\mathbf{y}) \right). \tag{38}$$

The formula (37) and (38) provide useful information on the manufacturing of f and g. For instance, suppose these surfaces are made from molds that we denote by F and G, respectively. Assume further that the molds F and G are cut by a CNC machine. The first step is to design the lens, i.e. to compute the surfaces f and g. This is done by the numerical procedure outlined above. The second step is to select a cutting tool with the shape $z = D^*(\mathbf{x})$ for the CNC machine. Since D^* is convex, such a tool is easy to make. Consider for example the upper

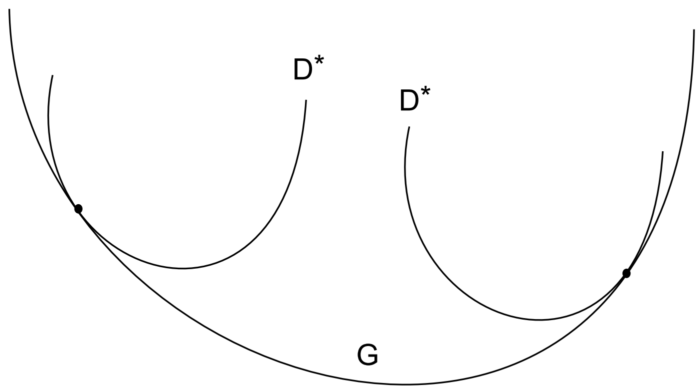

Figure 2: The manufacturing of the mold surface G by the cutting tool D^*.

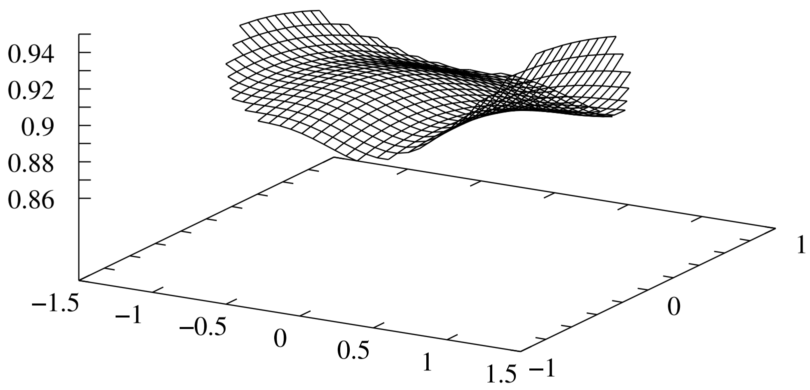

Figure 3: The target intensity I_2

lens surface g. Since we can identify the mold surfaces G and F with the negative of the lens surfaces g and f, respectively, we can use (37) and (38) to write

$$G(\mathbf{x}) = \min_{\mathbf{y}} \left(D^*(\mathbf{x} - \mathbf{y}) + F(\mathbf{y}) \right). \tag{39}$$

Equation (39) together with $\mathbf{y} = T^{-1}(\mathbf{x})$ is now interpreted as providing the shift $\mathbf{y}(\mathbf{x})$ and the elevation $F(\mathbf{y}(\mathbf{x}))$ for the position of the CNC tool relative to its rest position $D^*(\mathbf{x})$. The minimality property of (39) guarantees that the tool will cut exactly the desired shape G. The cutting process is sketched in Figure 2. The mold surface F can be cut by a similar method.

We give two examples for lenses designed by the method derived in this paper. In the first example we computed a lens that transforms a collimated beam with a uniform intensity into another collimated beam with the quite general profile I_2 depicted in Figure 3. Notice that the intensity I_2 has no symmetry, and in fact, is not even strictly convex or concave. The

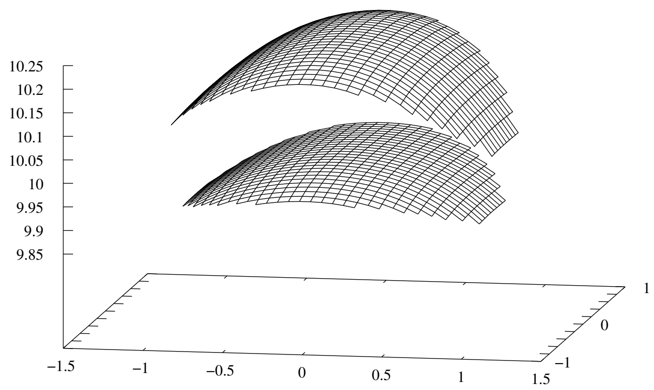

Figure 4: The shape of the two surfaces of a lens that shapes a uniform collimated beam into a collimated beam with the intensity distribution drawn in Figure 2

two surfaces of a lens that perform the required transformation are shown in Figure 4. In the current example both lens surfaces turned out to be convex (concave).

In the second example we considered the shaping of a hollow beam, namely, a beam whose intensity is supported on an annulus. The incident beam has the radial intensity profile

$$I_1(r) = \begin{cases} 1 + 1/4r & 1 \le r \le 5, \\ 0 & r < 1, \quad r > 5. \end{cases} \tag{40}$$

The outgoing beam has a uniform intensity $I_2 = 1$ over its support. The lens is computed first over the area that is reached by rays in the beams. It is then completed inside the 'hole' by requiring that it has a vertex at the origin and that it is smooth. A cross section of the lens is depicted in Figure 5

Finally, we make two comments:

1. Notice that the 'tool' $D^*(\mathbf{x})$ that was used in (39) to cut the surface g (or its mold) is universal in the sense that it does not depend on the intensities I_1 and I_2. Assuming that the refractive index is fixed, D^* only depends on the parameter χ. The actual intensities I_1 and I_2 enter the manufacturing process only through the elevation function, say, f in the manufacturing of g, and the shift value $\mathbf{y} = T^{-1}(\mathbf{x})$.

2. The design problem can be solved only if the functional $\mathbf{M}$ is finite for at least one feasible ray mapping T. To understand the limitation, assume that the support of the density I_2 is very far from the support of the density I_1. In this case, all the rays in any feasible mapping T satisfy the criterion $|T(\mathbf{x}) - \mathbf{x}| \geq \sqrt{\chi/bn}$ and of course $\mathbf{M}$ cannot

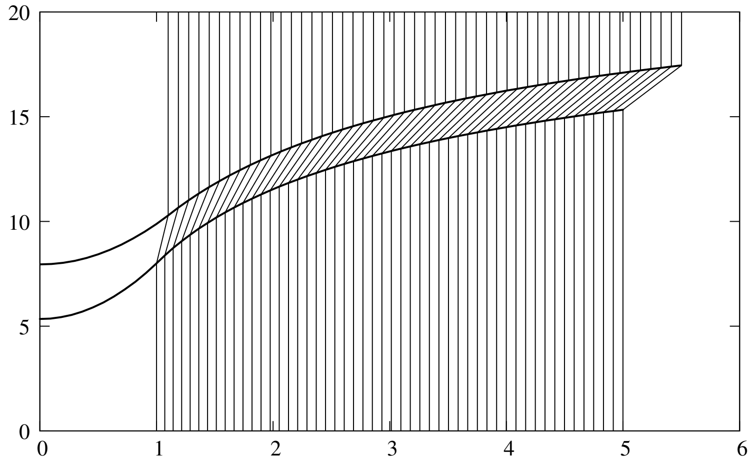

Figure 5: The shape of the two surfaces of a lens that converts a annular beam with linearly decreasing intensity into an annular uniform beam. The ray mapping is depicted as well. The lens is extended into the hole.

be minimized then. This is similar to the relativistic constraint on transport by the Klein-Gordon equation [6].

To summarize, we presented a method for the design of a refractive lens that shape arbitrary collimated beams. The lens surfaces are found by a solving a variational problem that is related to the weighted least action principle. The surfaces can be made by a milling machine with a convex tool.

5 Appendix A. The special case of beams with nearly the same intensity

Although we derived an explicit solution method, it involves a nontrivial optimization problem. In this appendix we consider the special case when I_1 is close to I_2. In this case we shall substitute the optimization problem by an elliptic partial differential equation. While the equation is nonlinear, it can be solved by well-known stable numerical methods.

Assume, then, that I_2 can be expressed in form

$$I_2(\mathbf{x}) = I_1(\mathbf{x})(1 + \alpha q(\mathbf{x})), \tag{41}$$

where $\alpha > 0$ is a small parameter. The correction function q satisfies the energy conservation condition $\int\int I_1(\mathbf{x})q(\mathbf{x})d\mathbf{x} = 0$.

Since the intensity I_1 is only slightly perturbed, we assume that the optimal ray mapping T is a perturbation of the identity mapping, that is

$$T(\mathbf{x}) = \mathbf{x} + \alpha\zeta(\mathbf{x}), \tag{42}$$

where ζ is a vector field that is to be determined. We expand

$$I_2(T(\mathbf{x})) = I_1(T(\mathbf{x}))(1 + \alpha q(T(\mathbf{x}))) = I_1 + \alpha\left[\nabla I_1 \cdot \zeta + I_1 q\right] + O(\alpha^2). \tag{43}$$

The substitution of equation (43) into equation (4) gives (up to $O(\alpha)$)

$$J(T) = \frac{1}{1 + \alpha\left[\nabla \ln(I_1) \cdot \zeta + q\right]} = 1 - \alpha\left[\nabla \ln(I_1) \cdot \zeta + q\right]. \tag{44}$$

On the other hand, the expansion of the determinant of $\nabla T = E_d + \alpha\nabla\zeta$, where E_d is the identity matrix, yields

$$J(T) = 1 + \alpha\nabla \cdot \zeta. \tag{45}$$

Comparing the two expressions (44) and (45) for J we obtain an equation for ζ:

$$\nabla \cdot \zeta + \nabla \ln(I_1) \cdot \zeta + q = 0. \tag{46}$$

Equation (46) for ζ is too general. In fact, the solution ζ is restricted similarly to the restriction on the vector field (δ, ε) that was analyzed in section 2. To find an explicit expression for the constraint satisfied by ζ, recall equation (26). Using (42) we can write

$$\frac{\zeta}{|\zeta|}\left(D^*\right)'(\zeta) = \nabla\Phi. \tag{47}$$

Since $\left(D^*\right)' > 0$ we obtain

$$\left(D^*\right)'(\zeta) = \left(D^*\right)'(|\zeta|) = |\nabla\Phi|.$$

The duality relation between D and D^* (expressed in equation (29)) implies

$$|\zeta| = D'\left(|\nabla\Phi|\right),$$

hence

$$\frac{|\zeta|}{\left(D^*\right)'(\zeta)} = \frac{D'(|\nabla\Phi|)}{|\nabla\Phi|}.$$

Thanks to equation (47),

$$\zeta = D'(|\nabla\Phi|)\frac{\nabla\Phi}{|\nabla\Phi|}. \tag{48}$$

In the specific case of D that we use here

$$\frac{D'(|\nabla\Phi|)}{|\nabla\Phi|} = \frac{\chi^{1/2}}{\sqrt{1+b|\nabla\Phi|^2}},$$

and therefore we obtain from (46) the following equation for Φ:

$$\nabla\cdot\left(\frac{\nabla\Phi}{\sqrt{1+b|\nabla\Phi|^2}}\right) + \frac{\nabla\ln(I_1)\cdot\nabla\Phi}{\sqrt{1+b|\nabla\Phi|^2}} + \chi^{-1/2}q = 0. \tag{49}$$

Equation (49) is a nonlinear elliptic partial differential equation for Φ. The equation can be written also in an alternative (somewhat more natural) form

$$\nabla\cdot\left(\frac{I_1\nabla\Phi}{\sqrt{1+b|\nabla\Phi|^2}}\right) + \chi^{-1/2}qI_1 = 0. \tag{50}$$

Typically one is interested in the case where I_1 has a finite support D. In this case, equation (50) is considered over the domain D. Since according to (41) I_2 has the same support D, ζ is tangential to the boundary, and therefore equation (50) is supplemented with the boundary condition $\partial_n\Phi = 0$, where n denotes the normal to the boundary.

References

[1] F.M. Dickey, "Laser beam shaping", OPN April 2003.

[2] F.M. Dickey and H.C. Holswade, *Laser Beam Shaping: Theory and Techniques,* (Marcel Dekker, 2000).

[3] J.B. Keller and R.M. Lewis, "Asymptotic methods for partial differential equations: The reduced wave equation and Maxwell's equations", *in Surveys in Applied Mathematics, Vol 1*, J.B. Keller et al. Eds., 1-82 (Plenum Press, 1993).

[4] J.B. Keller, "The Inverse scattering problem in geometrical optics and the design of reflectors", IRE Trans. on Antennas and Prop. **AP-7**, 146-149 (1959).

[5] T. Glimm and V. Oliker, "Optical design of two-reflector systems, the Monge-Kantorovich mass transfer problem and Fermat's principle", Indiana Univ. Math. J. **53**, 1255-1277 (2004).

[6] J. Rubinstein and G. Wolansky, "A weighted least action principle for dispersive waves", Annals of Physics **317**, 271-284 (2005).

[7] J. Rubinstein and G. Wolansky, “Reconstruction of optical surfaces from ray data”, Optical Review **8**, 281-283 (2001).

[8] N. Davidson, A.A. Freisem and E. Hasman, “Optical coordinate transformations”, Appl. Opt. **31**, 1067-1073 (1992).

[9] Y. Brenier, “Polar factorization and monotone rearrangement of vector-valued functions”, Comm. Pure Appl. Math. **64**, 375-417 (1991).

[10] J. Rubinstein and G. Wolansky, “A variational principle in optics”, J. Opt. Soc. Amer. A **21**, 2164-2172 (2004).

[11] S. Angenent, S. Haker and A. Tannenbaum, “Minimizing flows for the Monge-Kantorovich problem”, SIAM J. Math. Anal. **35**, 61-97 (2003).

[12] C. Lee, J. Lowengrub, J. Rubinstein and X. Zheng, “Phase reconstruction by the weighted least action principle”, J. of Optics A: Pure and Applied Optics, **8**, 279-289 (2006).

[13] C. Villani, *Topics in Optimal Transportation,* (American Mathematical Soceity, Providence, Rhode Island, 2003).

[14] L.C. Evans, *Partial differential Equations,* (American Mathematical Society, Providence, Rhode Island, 2002).