Limit Theorems for Optimal Mass Transportation and Applications to Networks

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Abstract

It is shown that optimal network plans can be obtained as a limit of point allocations. These problems are obtained by minimizing the mass transportation on the set of atomic measures of prescribed number of atoms.

1 Introduction

Optimal mass transportation was introduced by Monge some 200 years ago and is, today, a source of a large number of results in analysis, geometry and convexity.

Optimal Network Theory was recently developed. It can be formulated in terms of Monge-transport corresponding to some non-standard metrics. For updated references on optimal networks via mass transportation see [BS, BCM].

In this paper we restrict ourselves to the transport of a finite number of points. Consider N points $\{x_1, \dots x_N\}$ (sources) in a state space (say, $\mathbb{R}^k$), and another N points $\{y_1, \dots y_N\} \subset \mathbb{R}^k$ (sinks). For each source x_i we attribute a certain amount of mass $m_i \geq 0$. Similarly, $m_i^* \geq 0$ is the capacity attributed to the sink y_i, while

$$\sum_1^N m_i = \sum_1^N m_i^* > 0 \ .$$

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We denote this system by an atomic measure $\lambda := \lambda^+ - \lambda^-$ where

$$\lambda^+ = \sum_1^N m_i \delta_{x_i} \quad ; \quad \lambda^- = \sum_1^N m_i^* \delta_{y_i} \tag{1.1}$$

where $\delta_{(\cdot)}$ is the Dirac delta function.

The object is to transport the masses from the sources to sinks in an optimal way, such that the sinks are filled up according to their capacity. A natural cost was suggested by Xia [X]: For each $q > 1$, $\widehat{W}^{(q)}(\lambda)$ is defined below:

Definition 1.1. *Given λ as in (1.1),*

1. *An oriented, weighted graph $(\hat{\gamma}, m)$ associated with λ is a graph $\hat{\gamma}$ embedded in $\mathbb{R}^k$, composed of vertices $V(\hat{\gamma})$ and edges $E(\hat{\gamma})$. The* orientation of an edge $e \in E(\hat{\gamma})$ is determined by $\partial e = v_e^+ - v_e^-$ where $v_e^\pm \in V(\hat{\gamma})$ are the vertices composing the end points of e. The graph $\hat{\gamma}$ and the capacity function $m : E(\hat{\gamma}) \to \mathbb{R}^+ \cup \{0\}$ satisfy

   (a) $\{x_1, \dots x_N, y_1, \dots y_N\} \subset V(\hat{\gamma})$.

   (b) For each $i \in \{1, N\}$, $\sum_{\{e, x_i \in \partial^+ e\}} m_e = m_i$ and $\sum_{\{e, y_i \in \partial^- e\}} m_e = m_i^*$, where $\partial^\pm e := v_e^\pm$.

   (c) For each $v \in V(\hat{\gamma}) - \{x_1, \dots y_N\}$, $\sum_{\{e; v \in \partial^+ e\}} m_e = \sum_{\{e; v \in \partial^- e\}} m_e$.

2. The set of all weighted graphs associated with λ is denoted by $\Gamma(\lambda)$.

3. $$\widehat{W}^{(q)}(\lambda) := \inf_{(\hat{\gamma}, m) \in \Gamma(\lambda)} \sum_{e \in E(\hat{\gamma})} |e| m_e^{1/q} \tag{1.2}$$

There are two special cases which should be noted. In the limit case $q = 1$ the optimal graph satisfies $V(\hat{\gamma}) = \{x_1, \dots y_N\}$ and $\widehat{W}^{(1)}(\lambda) = W_1(\lambda^+, \lambda^-)$. Here $W_q(\lambda^+, \lambda^-)$ for $q \geq 1$ is the Wasserstein distance between λ^+ to λ^-,

$$W_q(\lambda^+, \lambda^-) := \left(\min_{\{\gamma^{i,j}\}} \sum_1^N \sum_1^N |x_i - y_j|^q \gamma_{i,j} \right)^{1/q} ,$$

the minimum is taken in the set of $N \times N$ matrices satisfying

$$\gamma_{i,j} \geq 0 \quad , \quad \sum_{i=1}^{N} \gamma_{i,j} = m_j^* \quad ; \quad \sum_{j=1}^{N} \gamma_{i,j} = m_i \ .$$

In particular, W_1 depends only on the difference $\lambda = \lambda^+ - \lambda^-$ (which is not the case for $q > 1$).

The second case is the limit $q = \infty$. This is the celebrated *Steiner Tree Problem* [HRW]:

$$\inf_{\hat{\gamma} \in \Gamma(\lambda)} \sum_{e \in E(\hat{\gamma})} |e| \ ,$$

where, this time, $\Gamma(\lambda)$ is the set of all graphs satisfying $\{x_i, y_j; m_i, m_j^* > 0\} \subset V(\hat{\gamma})$ and is, actually, independent of the masses m_i and capacities m_i^* (assumed positive).

In [W, Thm 2] it was shown that W_1 is obtained from W_q by an asymptotic expression for the limit of infinite mass:

Theorem 1.1. *If $\lambda = \lambda^+ - \lambda^-$ is any Borel measure satisfying $\int d\lambda = 0$, then*

$$\lim_{M \to \infty} M^{1-1/q} \min_{\mu \in \mathcal{B}_M^+} W_q \left(\mu + \lambda^+, \mu + \lambda^- \right) = W_1(\lambda^+, \lambda^-)$$

where $\mathcal{B}_M^+$ stands for the set of all positive Borel measures μ normalized by $\int d\mu = M$.

If, in particular, λ is an atomic measure of the form (1.1), than it can be shown that for fixed M the minimizer of $W_q\left(\mu + \lambda^+, \mu + \lambda^-\right)$ in $\mathcal{B}_M^+$ is an atomic measure of a finite number of atoms as well.

The main result of the current paper demonstrates that the network cost $\widehat{W}^{(q)}$ is obtained by similar expression, where the total mass M is replaced by the *cardinality of the support of the atomic measure μ.*

2 Main results

Here, and throughout the paper (excluding section 5), we assume that λ is an atomic measures with a *finite number of atoms*, as in (1.1). For each $n \in \mathbb{N}$, let $\mathcal{B}^{+,n}$ be the set of all atomic, positive measures of at most n atoms, that is:

$$\mathcal{B}^{+,n} := \left\{ \sum_{j=1}^{n} \alpha_j \delta_{z_j} \quad ; \quad z_j \in \mathbb{R}^k, \ \alpha_j \geq 0 \right\} .$$

Theorem 2.1. *For any $q > 1$*

$$\lim_{n\to\infty} n^{1-1/q} \inf_{\mu\in\mathcal{B}^{+,n}} W_q \left(\mu + \lambda^+, \mu + \lambda^-\right) = \widehat{W}^{(q)}(\lambda) \ . \tag{2.1}$$

The set $\mathcal{B}^{+,n}$ is, evidently, not a compact one. Still we claim

Lemma 2.1. *For each $n \in \mathbb{N}$, a minimizer $\mu_n \in \mathcal{B}^{+,n}$*

$$\overline{W}_q^{(n)}(\lambda^+, \lambda^-) := \inf_{\mu\in\mathcal{B}^{+,n}} W_q \left(\mu + \lambda^+, \mu + \lambda^-\right) = W_q \left(\mu_n + \lambda^+, \mu_n + \lambda^-\right) \tag{2.2}$$

exists.

Remark 2.1. *Note that $\overline{W}_q^{(n)}$ depends on each of the component $\lambda^\pm$ while the limit $\widehat{W}^{(q)} = \lim_{n\to\infty} n^{1-1/q}\overline{W}_q^{(n)}$ depends only on the difference $\lambda = \lambda^+ - \lambda^-$.*

Theorem 2.2. *Let μ_n be a regular[2] minimizer of $W_q (\mu + \lambda^+, \mu + \lambda^-)$ in $\mathcal{B}^{+,n}$. Then the associated optimal plan spans a reduced weighted tree[3] $(\hat{\gamma}_n, m_n)$ which converges (in Hausdorff metric) to an optimal graph $(\hat{\gamma}, m) \in \Gamma(\lambda)$ of (1.2) as $n \to \infty$,*

3 Auxiliary results

We first reformulate $\overline{W}_q^{(n)}$, as given by (2.2), in terms of a linear programming:

Given $q > 1$, $n \in \mathbb{N}$, $Z = (z_{N+1}, \dots z_{N+n}) \in (\mathbb{R}^k)^n$, $\lambda = \lambda^+ - \lambda^-$ as given by (1.1) and $\gamma := \{\gamma_{i,j} \ 1 \leq i, j \leq n + N\} \in \Gamma(n, \lambda^+, \lambda^-) :=$

$$\left\{ \gamma_{i,j} \geq 0 \ , \quad 1 \leq k \leq N \Longrightarrow \sum_{i=1}^{n+N} \gamma_{k,i} = m_k, \quad \sum_{i=1}^{n+N} \gamma_{i,k} = m_k^* \right.$$
$$\left. \sum_{i=1}^{n+N} \gamma_{i,j} = \sum_{i=1}^{n+N} \gamma_{j,i} \text{ for any } \ N+1 \leq j \leq n+N \right\} \ , \tag{3.1}$$

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[2] see Definition 3.1

[3] See Definitions 3.2 and 3.4

Let

$$F_q(Z,\gamma) := \sum_1^{n+N}\sum_1^{n+N} \gamma_{i,j} F_{i,j}(Z) \tag{3.2}$$

where

$$F_{i,j}(Z) := \begin{cases} |z_i - z_j|^q & N+1 \leq i,j \leq n+N \\ |x_i - z_j|^q & 1 \leq i \leq N, N+1 \leq j \leq n+N \\ |z_i - y_j|^q & 1 \leq j \leq N, N+1 \leq i \leq n+N \\ |x_i - y_j|^q & 1 \leq i,j \leq N \end{cases} \tag{3.3}$$

We observe

$$\overline{W}_q^{(n)}(\lambda^+,\lambda^-) \equiv \inf_{Z\in(\mathbb{R}^k)^n, \gamma\in\Gamma(n,\lambda^+,\lambda^-)} F_q(Z,\gamma) \ . \tag{3.4}$$

Our first object is to prove Lemma 2.1, that is, to replace the "inf" in (3.4) by "min".

Definition 3.1. *$\gamma \in \Gamma(n,\lambda^+,\lambda^-)$ is called a* regular plan *if it satisfies the following for any $1 \leq i,j \leq n+N$:*

(a) if $k \geq 1$ and $i_1 = i, i_2, \dots i_k = i$ then $\Pi_{l=1}^{k-1}\gamma_{i_l,i_{l+1}} = 0$. (In particular $\gamma_{i,j}\gamma_{j,i} = 0$ and $\gamma_{i,i} = 0$).

(b) If $k > 1$, $k' \geq 1$ and $\{i_1 = i, i_2, \dots i_k = j\} \not\equiv \{i'_1 = i, i'_2, \dots i'_{k'} = j\}$ then $\left(\Pi_{l=1}^{k-1}\gamma_{i_l,i_{l+1}}\right)\left(\Pi_{l=1}^{k'-1}\gamma_{i'_l,i'_{l+1}}\right) = 0$.

If γ is a regular plan, then $\mu \in \mathcal{B}^{+,n}$ is called a regular measure *if for each $i \in \{N+1,\dots n+N\}$ there exists $z_i \in \mathbb{R}^k$ where $\mu(\{z_i\}) = \sum_{j=1}^{n+N}\gamma_{i,j}$.*

Lemma 3.1. *For each $Z \in (\mathbb{R}^k)^n$ and any plan $\gamma \in \Gamma(n,\lambda^+,\lambda^-)$ there exists a regular plan $\gamma^r \in \Gamma(n,\lambda^+,\lambda^-)$ satisfying $F_q(Z,\gamma^r;) \leq F_q(Z,\gamma)$.*

Proof. a) Assume $\Pi_{l=1}^{k-1}\gamma_{i_l,i_{l+1}} > 0$. Let i_{l_0} such that $\gamma_{i_{l_0},i_{l_0}+1} \leq \gamma_{i_l,i_l+1}$ for any $1 \leq l < k$. Then $\gamma^{r_1}_{i_l,i_l+1} := \gamma_{i_l,i_l+1} - \gamma_{i_{l_0},i_{l_0}+1}$ while $\gamma^{r_1}_{i,j} = \gamma_{i,j}$ otherwise. It follows that $\gamma^{r_1} \in \Gamma(n,\lambda^+,\lambda^-)$ and $F_q(Z,\gamma^{r_1}) \leq F_q(Z,\gamma)$. Thus γ^{r_1} verifies (a).

b) We may assume that $\{i_2, \dots i_{k-1}\} \cap \{i'_2, \dots i'_{k'-1}\} = \emptyset$ for, otherwise, choose 2 pairs of indices $i_l = i'_{l'}$ and $i_m = i'_{m'}$ for which $\{i_{l+1}, \dots i_{m-1}\} \cap \{i'_{l'+1}, \dots i'_{m'-1}\} = \emptyset$. Assume $\left(\Pi_{l=1}^{k-1} \gamma^{r_1}_{i_l, i_{l+1}}\right) \left(\Pi_{l=1}^{k'-1} \gamma^{r_1}_{i'_l, i'_{l+1}}\right) > 0$. Assume (with no limitation to generality) that $\sum_{l=1}^{k-1} |Z_{i_l} - Z_{i_{l+1}}|^{1/q} \geq \sum_{l=1}^{k'-1} |Z_{i'_l} - Z_{i'_{l+1}}|^{1/q}$. Let i_{l_0} such that $\gamma^{r_1}_{i_{l_0}, i_{l_0}+1} \leq \gamma^{r_1}_{i_l, i_l+1}$ for any $1 \leq l < k$. Then set

$$\gamma^r_{i'_l, i'_{l+1}} := \gamma^{r_1}_{i'_l, i'_{l+1}} + \gamma^{r_1}_{i_{l_0}, i_{l_0}+1}$$

$$\gamma^r_{i_l, i_{l+1}} := \gamma^{r_1}_{i_l, i_{l+1}} - \gamma^{r_1}_{i_{l_0}, i_{l_0}+1}$$

while $\gamma^r_{i,j} = \gamma^{r_1}_{i,j}$ otherwise. Then γ^r verifies (3.1) while

$$F_q(Z, \gamma^r) = F_q(Z, \gamma^{r_1}) - \gamma^{r_1}_{i_{l_0}, i_{l_0}+1} \left[\sum_{l=1}^{k-1} |Z_{i_l} - Z_{i_{l+1}}|^{1/q} - \sum_{l=1}^{k'-1} |Z_{i'_l} - Z_{i'_{l+1}}|^{1/q} \right]$$
$$\leq F_q(Z, \gamma^{r_1}) \leq F_q(Z, \gamma) \quad (3.5)$$

□

Lemma 3.2. *The set of regular plans in $\mathcal{B}^{+,n}$ associated with $\Gamma(n, \lambda^+, \lambda^-)$ (3.2) is compact.*

Proof. Let z_i be some point in the support of μ where $\mu(\{z_i\}) = Q$. We show an a-priori bound on Q (hence compactness). By (3.1) there exists a point z_{i_2} where $\gamma_{i,i_2} \geq Q/(N+n)$. We can define such a chain $i = i_1, i_2, \dots$ where $\gamma_{i_l, i_{l+1}} > \mu(\{z_{i_l}\})/(n+N)$. In particular it follows that $\mu(\{z_{i_l}\}) \geq Q/(n+N)^{l-1}$. By part (a) of the definition of regular plans, this chain must be of length ar most n. By (3.1) it must end at some $i_k := j \in \{1, \dots N\}$. So, $\mu(\{z_j\}) \geq Q/(n+N)^{k-1} \geq Q/(n+N)^{n-1}$. On the other hand, $\mu(\{z_j\}) \leq \max_{1 \leq l \leq N} \max\{m_l, m^*_l\} := M$ so $Q \leq M(n+N)^{n-1}$. □

Corollary 3.1. *For fixed $Z \in (\mathbb{R}^k)^n$, λ satisfying (1.1) and $q > 1$, the function F_q admits a minimizer $\gamma \in \Gamma(n, \lambda^+, \lambda^-)$. Moreover, this minimizer is regular.*

Proof. of lemma 2.1

For a fixed λ satisfying (1.1) and $q > 1$ it follows from Corollary 3.1 that

$$\overline{F}_q(Z,\gamma) := \inf_{\gamma\in\Gamma(n,\lambda^+,\lambda^-)} F_q(Z,\gamma) = \min_{\gamma\in\Gamma(n,\lambda^+,\lambda^-)} F_q(Z,\gamma) \ .$$

It is also evident that $\overline{F}_q$ is continuous and coercive on $(\mathbb{R}^k)^n$ and that $\overline{W}_q^{(n)}(\lambda^+,\lambda^-) = \min_{Z\in(\mathbb{R}^k)^n} \overline{F}_q(Z,\lambda)$. In particular (3.4) is attained at a pair (Z,γ) where

$$\mu_n := \sum_{i,j=N+1}^{n+N} \gamma_{i,j}\delta_{z_i} \in \mathcal{B}^{+,n} \tag{3.6}$$

is a regular minimizer of (2.2). □

Next we associate a weighted graph $(\hat{\gamma}, m)$ with a transport plan $\gamma \in \Gamma(n,\lambda^+,\lambda^-)$ and $Z \in (\mathbb{R}^k)^n$ as follows (see Fig 1)

Definition 3.2. *Let $Z = \{z_{N+1},\dots z_{N+n}\} \in (\mathbb{R}^k)^n$ and $\gamma \in \Gamma(n,\lambda^+,\lambda^-)$. The associated weighed, directed graph $(\hat{\gamma}, m)$ is defined as*

(i) $V(\hat{\gamma}) = \{x_1,\dots y_N, z_1,\dots z_n\} := \{\zeta_1,\dots \zeta_{n+2N}\}$.

(ii) $E(\hat{\gamma})$ is given by the set of segments $e_{k,l} := [\zeta_k,\zeta_l]$ for which $\gamma_{k,l} > 0$, while $\partial e_{k,l} = \zeta_k - \zeta_l$.

(iii) $m_{e_{k,l}} := \gamma_{k.l}$.

(iv) For each $z_i \in V(\hat{\gamma})$, $deg(z_i) := \#\{j; \gamma_{i,j} + \gamma_{j,i} > 0\}$.

Lemma 3.3. *Let (Z,γ) as in Definition 3.2 where γ is a regular plan in $\Gamma(\lambda, n)$. Then the associated graph $(\hat{\gamma}, m)$ contains no cycles. In addition, $|E(\hat{\gamma})| \le n + 2N^3$.*

Proof. The result that the graph $\hat{\gamma}$ contains no cycles follows directly from Definition 3.1-a.

It follows that any vertex $v \in V(\hat{\gamma})$ must belong to a chain $C_{i,j} := \zeta_1,\dots \zeta_k$ where $k \le n$, $\zeta_1 = x_i$ and $\zeta_k = y_j$. By Definition 3.1-b there exists at most one such a chain

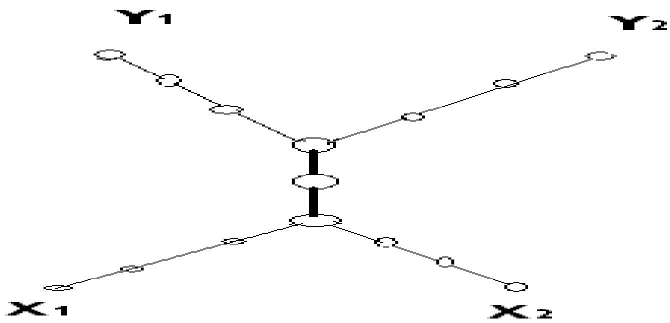

Figure 1: A tree associated with a regular transport plan ($N = 2$, $n = 11$)

for any pair $(x_i, y_j) \in \{x_1, \dots x_N\} \times \{y_1, \dots y_N\}$. In particular there exists at most N^2 such chains.

Let now $\zeta_l \in C_{i,j}$. If the degree of ζ_l is greater than 2, there exist $deg(\zeta_l) - 1 > 1$ chains which contain ζ_l. By Definition 3.1-b it follows that if two chains $C_{i',j'}$, $C_{i",j"}$ intersect the chain $C_{i,j}$ then either $C_{i',j'} = C_{i",j"}$ (and, in particular, they intersect $C_{i,j}$ at the same point), or $i" \neq i'$ and $j" \neq j'$. Hence the number of chains crossing $C_{i,j}$ is bounded by $2N$. As the number of chains $\{C_{i,j}\}$ is bounded by N^2 it follows that there exists at most $2N^3$ chains which intersect other chains. Hence $\sum_{v \in V(\hat{\gamma})}(deg(v) - 2) \leq 2N^3$ which implies the result. □

Next, we elaborate some properties of an optimal regular plan.

Definition 3.3. *A chain of a regular plan is a sequence of indices* $i_1, \dots, i_k$ *such that* $\gamma_{i_l, i_{l+1}} > 0$ *for* $k > l \geq 1$ *while* $\gamma_{i_l, j} = 0$ *for any* $j \in \{1 \dots, n + 2N\}$. *A maximal chain is a chain which is not contained in a larger chain.*

Remark 3.1. *By (3.1) we also get that* $\gamma_{i_l, i_{l+1}}$ *is a constant along any maximal chain* $i_1, \dots, i_k$ *where* $1 < l < k$.

Lemma 3.4. *If* γ *is a regular* optimal *plan then for any chain* $\{\zeta_{i_1}, \dots \zeta_{i_k}\}$, $\zeta_{i_{l+1}} - \zeta_{i_l} = \zeta_{i_{l'+1}} - \zeta_{i_{l'}}$ *for any* $l, l' \in \{1, \dots k - 1\}$. *In particular, all points on a chain of the*

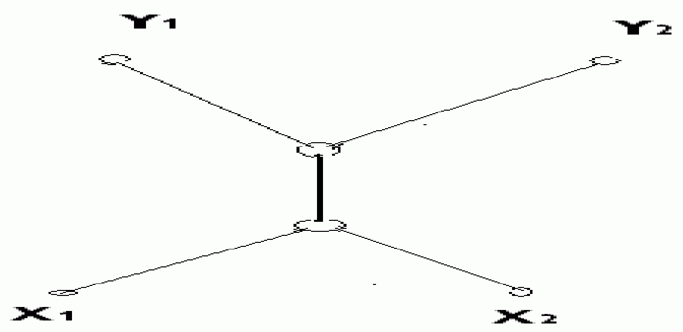

Figure 2: The reduced version of the tree presented in Fig 1: All vertices of degree 2 removed.

associated directed graph corresponding to an optimal plan are equally spaced on a line segment*.*

Proof. If γ_0^R is a regular optimal plan then $Z = (z_1, \ldots z_n)$ is a minimizer of $F_q(Z, \gamma_0^R)$ in $(\mathbb{R}^k)^n$. In particular $\frac{\partial F_q}{\partial z_j} = 0$ holds for any $1 \leq j \leq n$. If $j = i_l$ is embedded in a chain then by definition and Remark 3.1 we obtain

$$\frac{\partial F_q}{\partial z_j} = q\gamma_{i_l, i_{l+1}} \left[\frac{z_{i_l} - z_{i_{l-1}}}{|z_{i_l} - z_{i_{l-1}}|^{q-2}} - \frac{z_{i_{l+1} - z_{i_l}}}{|z_{i_{l+1}} - z_{i_l}|^{q-2}} \right] = 0$$

which implies the result. □

Let us now re-define the associated directed graph $(\hat{\gamma}, m)$ corresponding to *an optimal* regular plan (see Fig 2)

Definition 3.4. *The* reduced *weighted graph* $(\hat{\gamma}_R, m)$ *associated with an optimal regular plan is obtained from* $(\hat{\gamma}, m)$ *(Definition 3.2) by identifying all edges corresponding to a maximal chain* $\{i_1, \ldots i_k\}$ *with a single edge* $[\zeta_{i_1}, \zeta_{i_k}]$ *and assigning the the common weight* $m_e = \gamma_{i_l, i_{l+1}}$ *to this edge (recall Remark 3.1).*

Corollary 3.2. *A reduced weighted graph* $(\hat{\gamma}_R^n, m)$ *associated with an optimal regular plane in* $\mathcal{B}^{+,n}$ *satisfies the following:*

(i) All the vertices of $\hat{\gamma}_R^n$ are of degree at least 3.

(ii) The number of vertices of $\hat{\gamma}_R^n$ is at most $2N^3$ where N is the number of atoms of $\lambda^{\pm}$ (in particular, independent of n).

(iii) All the edges of $\hat{\gamma}_R^n$ are line segments.

(iv) There exists $C > 0$, depending only on N, such that $C > m_e > 1/C$ for any $e \in E(\gamma_R^n)$.

(v) There is a compact set $K \subset \mathbb{R}^k$ which contains $\hat{\gamma}_R^n$ for any $n \in \mathbb{N}$.

Proof. Part (i) follows directly from Definition 3.4. Part (ii) from Lemma 3.3, part (iii) from Lemma 3.4. To prove part (iv) we repeat the proof of Lemma 3.2, with the additional information of (ii) (that is, the bound on the number of edges is independent of n). Part (v) is evident. □

4 Proof of Theorems 2.1 and 2.2

Proof. of theorem 2.1:

Let $(\hat{\gamma}, m)$ be a weighted graph. Then by the Hölder inequality

$$\sum_{e\in E(\hat{\gamma})} m_e^{1/q}|e| \leq \left(\sum_{e\in E(\hat{\gamma})} m_e|e|^q \right)^{1/q} |E(\hat{\gamma})|^{(q-1)/q} . \tag{4.1}$$

If, moreover, $(\hat{\gamma}, m)$ is obtained from a regular plan $\gamma \in \Gamma(n, \lambda^+, \lambda^-)$ then

$$W_q^q(\mu_n + \lambda^+, \mu_n + \lambda^-) \leq \sum_{e\in E(\hat{\gamma})} m_e|e|^q \tag{4.2}$$

where $\mu_n \in \mathcal{B}^{+,n}$ associated with γ via (3.6). By Lemma 2.1 there exists an optimal measure $\mu_n \in \mathcal{B}^{+,n}$. Hence (4.2) holds with an equality for this choice of μ_n. Moreover, μ_n can be chosen to be a regular measure (Definition 3.1) hence, by (4.1,4.2) and by

Lemma 3.3

$$\widehat{W}_q(\lambda) \le \sum_{e\in E(\hat\gamma)} m_e^{1/q}|e| \le \overline{W}_q^{(n)}(\lambda^+,\lambda^-)|n+2N^3|^{(q-1)/q} \ .$$

This implies the inequality

$$\liminf_{n\to\infty} n^{1-1/q}\overline{W}_q^{(n)}\left(\lambda^+,\lambda^-\right) \ge \widehat{W}^{(q)}(\lambda) \ .$$

To prove the reverse inequality in (2.1) we consider an optimal weighed graph $(\hat\gamma, m)$ of $\widehat{W}_q(\lambda)$ and construct $\mu_n \in \mathcal{B}^{+,n}$ supported on $\hat\gamma$ which satisfy

$$\lim_{n\to\infty} n^{1-1/q}W_q\left(\mu_n+\lambda^+,\mu_n+\lambda^-\right) = \sum_{e\in E(\hat\gamma)} m_e^{1/q}|e| = \widehat{W}^{(q)}(\lambda) \ .$$

Assume n_e is the number of points of μ_n on the edge e, and any atom of μ_n in e is of weight m_e. The contribution to $W_q^q(\mu_n+\lambda^+,\mu_n+\lambda^-)$ from e is, then

$$\approx m_e\left(\frac{|e|}{n_e}\right)^q n_e = \frac{m_e|e|^q}{n_e^{q-1}}$$

$$n^{q-1}W_q^q(\mu_n+\lambda^+,\mu_n+\lambda^-) \approx n^{q-1}\sum_{e\in E(\hat\gamma)}\frac{m_e|e|^q}{n_e^{q-1}}$$

The constraint on n_e is given by $\sum_{e\in E(\hat\gamma)} n_e = n$. Let us rescale $w_e := n_e/n$. Then we need to minimize

$$F(w) := \sum_{e\in E(\hat\gamma)}\frac{m_e|e|^q}{w_e^{q-1}}$$

subjected to $\sum_{e\in E(\hat\gamma)} w_e = 1$. Let α be the Lagrange multiplier with respect to the constraint $\sum_{e\in E(\hat\gamma)} w_e$. Since F is convex in w_e we get that F is maximized at

$$\max_\alpha \min_w F(w) + \alpha(\sum_{e\in E(\hat\gamma)} w_e - 1) \ . \tag{4.3}$$

So. let

$$G(\alpha) := \min_w F(w) + \sum_{e\in E(\hat\gamma)} w_e\alpha$$

The minimizer is obtained at

$$(q-1)\frac{m_e|e|^q}{w_e^q} = \alpha \Longrightarrow w_e = (q-1)^{1/q} m_e^{1/q} |e| \alpha^{1/q}$$

so

$$G(\alpha) = q(q-1)^{1/q-1} \sum_{e\in E(\hat{\gamma})} m_e^{1/q}|e|\alpha^{(q-1)/q} \ .$$

and the minimum is obtained at

$$\min_{(m,\hat{\gamma})\in\Gamma(\lambda)} \max_{\alpha} G(\alpha) - \alpha = \max_{\alpha} q(q-1)^{1/q-1}\widehat{W}^{(q)}(\lambda)\alpha^{(q-1)/q} - \alpha = \left(\widehat{W}^{(q)}(\lambda)\right)^q \quad (4.4)$$

□

Proof. of Theorem 2.2:

Let us consider the sequence of reduced weighted graphs $(\hat{\gamma}_R^n, m_n)$ (see Definition 3.4) associated with a regular minimizer γ_n. By Corollary 3.2-(v) there exists a limit $\hat{\gamma}_R$ (in the sense of Hausdorff metric) of a subsequence of $\hat{\gamma}_R^n$. By (ii-iv) of the Corollary, $|E(\hat{\gamma}_R)| < 2N^3$ and is $E(\hat{\gamma}_R)$ is composed of lines. Moreover, the weights $m_n : E(\hat{\gamma}_R^n) \to \mathbb{R}^+$ converges also, along a subsequence, to $m : E(\hat{\gamma}_R) \to \mathbb{R}^+$ so $(m, \hat{\gamma}_R) \in \Gamma(\lambda)$ (see Definition 1.1-(2)). Moreover

$$\lim_{n\to\infty} \sum_{e\in E(\hat{\gamma}_R^n)} m_{n,e}^{1/q}|e| = \sum_{e\in E(\hat{\gamma}_R)} m_e^{1/q}|e| \quad (4.5)$$

By definition of the reduced graph (Definition 3.4) and, in particular, Remark 3.1 we observe that $\sum_{e\in E(\hat{\gamma}_R^n)} m_{n,e}^{1/q}|e|$ is identical to the same expression on the non reduced graph $\hat{\gamma}^n$:

$$\sum_{e\in E(\hat{\gamma}_R^n)} m_{n,e}^{1/q}|e| = \sum_{e\in E(\hat{\gamma}^n)} m_{n,e}^{1/q}|e| \ . \quad (4.6)$$

However, on the non-reduced graphs we also have the inequalities (4.1, 4.2)

$$\sum_{e\in E(\hat{\gamma}^n)} m_{n,e}^{1/q}|e| \leq \left(\sum_{e\in E(\hat{\gamma}^n)} m_{n,e}|e|^q\right)^{1/q} |E(\hat{\gamma}^n)|^{(q-1)/q} = \overline{W}_q^{(n)}(\lambda^+,\lambda^-)|E(\hat{\gamma}^n)|^{(q-1)/q} \quad (4.7)$$

where $\overline{W}_q^{(n)}$ as defined in (2.2). Here we also used the optimality of γ^n.

Finally, from Theorem 2.1

$$\lim_{n\to\infty} \overline{W}_q^{(n)}(\lambda^+,\lambda^-)|E(\hat{\gamma}^n)|^{(q-1)/q} = \lim_{n\to\infty} \overline{W}_q^{(n)}(\lambda^+,\lambda^-)n^{(q-1)/q} = \widehat{W}_q(\lambda) \ .$$

This and (4.5-4.7) yields

$$\sum_{e\in E(\hat{\gamma}_R)} m_e^{1/q}|e| \leq \widehat{W}_q(\lambda)$$

while the opposite inequality follows from the definition of $\widehat{W}_q$. □

5 Open problems

There are many possible open problem related to the cost $\widehat{W}^q$ and the generalization of Theorems 2.1 and 2.2. Here I specify two of them

1. In the case where λ is an atomic measure whose support is countable. I conjecture that Theorems 2.1 and 2.2 still hold.

2. The case λ is not an atomic measure in $\mathbb{R}^k$. For example, if λ is continuous with respect to Lebesgue measure in $\mathbb{R}^k$, then [X] states that $\widehat{W}^q(\lambda) < \infty$ if $k/(k-1) > q > 1$. I conjecture that , in that case, Theorems 2.1 takes the form

$$\lim_{n\to\infty} n^{1/k} \inf_{\mu\in\mathcal{B}^{+,n}} W_q\left(\mu+\lambda^+,\mu+\lambda^-\right) = \widehat{W}^{(q)}(\lambda) \ .$$

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