

“ORIGAMI” PROOFS OF IRRATIONALITY OF SQUARE ROOTS

Michael Cwikel

Version 2.8 (22.8.2010). This is essentially the same as 2.7 of 7.8. 2007. The main change is that the reference [1] has been added and been connected with what is done here

The starting point of this document is the delightful proof by “paper folding” that $\sqrt{2}$ is irrational.

If somehow you have not yet seen that proof, please ask someone, for example me, to show it to you. (Or maybe look at [2]. Well, meanwhile you can also extract it as a special case (substitute $a = b$ and then $n = 1$) of what appears below in Section 2.) I learned this proof from Yoram Sagher who had learned it from John Conway. While we can consider it merely as a geometric realization of one of the algebraic proofs (see the appendix (Section 3)) of the irrationality of $\sqrt{2}$, some people have claimed that it might have been first discovered in antiquity by Indian or Japanese mathematicians.

The paper folding proof is of course longer than the algebraic proofs, both the usual one and the one given here in the appendix, but it seems to be universally agreed that it is much more fun. Furthermore it is perhaps more “real” for children, and thus it can be a beautiful introduction to mathematics for, say, eighth graders.

Acknowledgements. Thanks very much Yoram for telling me about that proof, and for many very interesting and thought provoking comments about this and related topics, before and since. Thanks also to many other people, ranging from some of the world’s most distinguished mathematicians, through to very motivated amateurs, and keen school pupils, who have enthusiastically joined me on various occasions in the pleasure of admiring and sharing this little gem.

This document is a humble attempt to somehow go beyond that proof.

My aim here is of course not to discover any new facts about the irrationality of various square roots, but rather to try to extend the above mentioned fun a bit further.

In Section 1 (essentially the same as it was in October 2003) of this document it is shown that $\sqrt{n^2 + 4}$ is irrational for every positive integer n . As kindly pointed out to me in 2000 by Jan Haugland, there is also a different and also genuine “paper folding” proof that $\sqrt{5}$ is irrational which works by successively folding a sequence of rectangles whose sides are in the ratio $2 : (\sqrt{5} + 1)$.

The folding performed in Section 1 is of an isosceles triangle. Since $n^2 + 4 > 2$ we cannot, strictly speaking, consider this argument as a generalization of the above mentioned proof of the irrationality of $\sqrt{2}$. However Section 2 of this document (written in July 2007) is indeed such a generalization. It gives a proof by paper folding of a right-angled triangle, that $\sqrt{n^2 + 1}$ and $\sqrt{n^2 - 1}$ are both irrational for every positive integer n . It is independent of Section 1 and perhaps you should read it before Section 1. As already indicated, Section 3 is an appendix.

It is intriguing to remark that we thus have three apparently different paper folding proofs of the irrationality of $\sqrt{5}$.

I would presume that everything here is already very well known, somewhere. If you know some references please tell me. Meanwhile Steven Miller kindly informed me of his joint paper [1] which gives other intriguing geometric approaches, via consideration of areas of various polygons, to proving irrationality of square roots of integers.

I may update this document from time to time. At any given moment the latest version of it will be posted at

<http://www.math.technion.ac.il/~mcwikel/paperfold.pdf>

1. A “paper folding” proof that $\sqrt{n^2 + 4}$ is irrational for every positive integer n .

Let’s first do the algebraic proof which motivates the “paper-folding” proof.

If $\sqrt{n^2 + 4}$ is rational then also $x = \frac{n + \sqrt{n^2 + 4}}{2}$ is rational and we shall write it in the form $x = p_1/q_1$ where p_1 and q_1 are positive integers. Note also that x must satisfy the equation $x = \frac{1}{x-n}$. Thus

$$\frac{p_1}{q_1} = \frac{1}{\frac{p_1}{q_1} - n} = \frac{q_1}{p_1 - nq_1}.$$

Since $x > 1$ we have that $q_1 < p_1$ and also $0 < p_1 - nq_1 < q_1$. Let us define $p_2 = q_1$ and $q_2 = p_1 - nq_1$. Then we can also write x in the form $x = p_2/q_2$. Reiterating this calculation we obtain two strictly decreasing infinite sequences of positive integers $\{p_m\}$ and $\{q_m\}$ which satisfy $x = p_m/q_m$ for each $m \in \mathbb{N}$, by setting $p_m = q_{m-1}$ and $q_m = p_{m-1} - nq_{m-1}$. This of course is impossible.

Now let us try to make this same argument look like “paper-folding” of triangles, or almost so.

Given a positive integer n let us suppose that $\sqrt{n^2 + 4}$ is rational. Then let $T(n)$ be the family of all isosceles triangles ABC whose sides all have integer lengths and which satisfy $AB = AC = x \cdot BC$ where $x = \frac{n + \sqrt{n^2 + 4}}{2}$. Since we are supposing that x is rational there must exist at least one triangle in $T(n)$. Let us denote it by T_1 and let its vertices be A_1, B_1 and C_1 , where $A_1B_1 = A_1C_1 = p_1$. Let us also denote the length B_1C_1 by q_1 . We now describe a procedure for constructing a strictly smaller triangle $T_2 \in T(n)$. It can be proved to be the same as folding in the special case where $n = 1$. Otherwise it is a little more complicated. Construct a line segment which is n times the length of B_1C_1 and use it to mark a point D_1 on A_1B_1 which is that distance from A_1 . Now set $A_2 = C_1, B_2 = B_1$ and $C_2 = D_1$. We claim that $A_2B_2 = A_2C_2$ and that the new triangle T_2 whose vertices are A_2, B_2 and C_2 is in $T(n)$. Our construction automatically gives that the lengths of $A_2B_2 = p_2$ and $B_2C_2 = q_2$ are integers and that the angle $\angle A_2B_2C_2 = \angle A_1B_1C_1$. If we can show that $p_2/q_2 = p_1/q_1$ then this will imply that T_2 and T_1 are similar triangles from which it will immediately follow that $A_2C_2 = p_2$ and so $T_2 \in T(n)$.

It is hard to make the remaining step, i.e. the proof that $p_2/q_2 = p_1/q_1$, look like geometry. It is just the same algebraic argument that we already gave above. Anyway here it is again: By construction we have $p_1/q_1 = x$ and $p_2 = q_1$ and $q_2 = p_1 - nq_1$. So we have to show that

$$\frac{q_1}{p_1 - nq_1} = \frac{p_1}{q_1}$$

which is the same as

$$\frac{1}{p_1/q_1 - n} = \frac{p_1}{q_1}$$

or, equivalently,

$$\frac{1}{x - n} = x$$

which is of course satisfied by our $x = \frac{n + \sqrt{n^2 + 4}}{2}$.

Where does this come from?

I am very grateful to Robert Langlands, who, in 2000, suggested to me that there ought to be a connection between geometric proofs of irrationality of certain numbers and the

representation of those numbers as infinite continued fractions. He referred me to Hardy and Wright's "An Introduction to the Theory of Numbers." I looked at the fourth edition (though there is a later edition.) Section 4.6 presents geometrical proofs of the irrationality of $\sqrt{2}$ and $\sqrt{5}$ which are apparently equivalent to the ones using triangles, but I think less attractive. The section begins by stating: "The proofs suggested by Zeuthen vary from number to number, and the variations depend at bottom on the form of the periodic continued fraction which represents \sqrt{N}"

Chapter 10 of the same book presents, among other things, the result that every irrational number x can be uniquely expressed as an infinite continued fraction

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

where all the numbers a_m are integers and $a_m > 0$ for $m = 1, 2, \dots$

Furthermore x is a (necessarily irrational) root of a quadratic equation with integer coefficients if and only if the sequence $\{a_m\}$ is ultimately periodic, i.e. for some fixed integers $k > 0$ and L it satisfies $a_{m+k} = a_m$ for all $m \geq L$.

The particular numbers with which we are dealing with above correspond to the special case where $L = 1$ and $k = 1$ and n is simply the common value of all the numbers a_m .

I cannot see at this moment how to generalize the arguments given above to the case when $k > 1$. Possibly the transition from p_m and q_m to p_{m+1} and q_{m+1} passes via k separate steps.

2. A "paper folding" proof that $\sqrt{n^2 + 1}$ and $\sqrt{n^2 - 1}$ are both irrational for every positive integer n .

Yes, I know, $\sqrt{1^2 - 1}$ is rational. But, anyway.

We are going to fold right angled triangles.

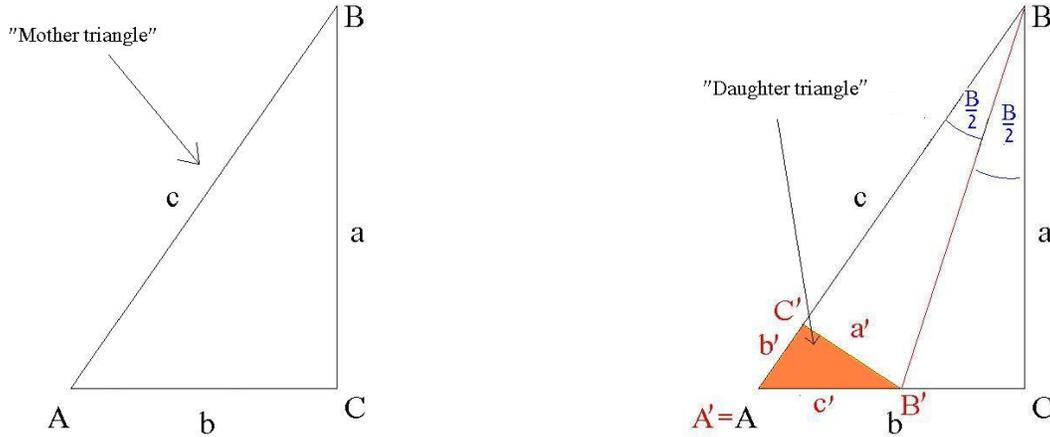
Here is a suggestion. You presumably have already seen the paper folding proof for $\sqrt{2}$. What I am going to do here is a rather natural generalization of it. You will almost certainly derive more enjoyment from discovering this generalization for yourself, than from reading the following stuff. The only ingredients you will need are some basic facts about similar triangles and Pythagoras' theorem ¹.

So, I see you have decided to look at my stuff anyway. Well, let ABC be an arbitrary right angled triangle. Let C be the vertex which is a right angle. We will use the customary notation a , b and c denote the lengths of the sides of the triangle opposite the vertices A , B and C respectively.

Starting with a particular right angled triangle ABC , which we will call the "mother triangle" we want to produce a new smaller right angled triangle $A'B'C'$, to be called the "daughter triangle" by "paper folding". (See the pictures below.) The triangles ABC and $A'B'C'$ will be similar to each other. As is done in the paper folding proof of the irrationality of $\sqrt{2}$, our act of folding is to place the side BC over the side AB . Even if we do not immediately realize it while we are folding, what we have really done is to first choose a

¹ It is believed (and mentioned for example on the web site of the Department of Mathematics of Haifa University) that Pythagoras lived for some time on Mount Carmel. So perhaps Pythagoras was once rather less than an hour's walk from the place from where I am now writing these lines.

point B' on the line AC such that the line BB' bisects the angle of the mother triangle at the vertex B . In fact we will “fold” our mother triangle along the line BB' . Thus, after folding, the point C will touch the line AB at a point which we will call C' .



We have specified two of the vertices B' and C' of the daughter triangle. We still have to specify one more vertex, the point A' . We will simply take $A' = A$.

The angle at the vertex C' of the daughter triangle is of course a right angle. Its angle at A' is of course the same as the angle at A of the “mother” triangle ABC . So its angle at B' must be the same as the angle at B in the “mother” triangle. Of course we will denote the lengths of the sides of the daughter triangle opposite its vertices A' , B' and C' by a' , b' and c' respectively.

Obviously we have $b' = c - a$ and $c' = b - a'$.

Remark. In the “classical” special case of the paper folding proof for $\sqrt{2}$ we have $a = b$ and therefore $a' = b'$, and therefore

$$(1) \quad c' = 2a - c \text{ and } a' = c - a.$$

Thus if we know that $\sqrt{2} = c/a$ it follows by “paper folding” that $\sqrt{2} = (2a - c)/(c - a)$. An alternative geometric proof of this same implication can be given by comparing the areas of various squares. This is exactly what is done in the proof of Stanley Tennenbaum, presented on page 2 of [1]. And of course this same implication can also be shown algebraically, very very easily. See Section 3.)

Theorem 2.1. *Suppose that all of the sides a , b and c of the “mother” triangle are integers, and that either $a = nb$ or $c = nb$ for some positive integer n . Then all the sides a' , b' and c' of the “daughter” triangle are also integers.*

Proof. Obviously b' is an integer.

If $a = nb$ for some positive integer, then it follows that $a' = nb'$ so that a' is also an integer. Consequently, $c' = b - a'$ is also an integer.

Alternatively, if $c = nb$ for some positive integer n , then it follows that $c' = nb'$ so that c' is an integer. This implies in turn that $a' = b - c'$ is an integer. \square

Corollary. $\sqrt{n^2 + 1}$ is irrational for every $n \in \mathbb{Z}$ and $\sqrt{n^2 - 1}$ is irrational for every positive integer $n > 1$.

Proof. Suppose first that $\sqrt{n^2 + 1}$ is rational. Set $\sqrt{n^2 + 1} = c/a$, where c and a are both integers. We consider the right angled triangle whose sides have lengths a , $b := na$ and

$\sqrt{a^2 + n^2 a^2} = c$. This will be our “mother” triangle. Thus we can apply Theorem 2.1 to obtain a similar smaller “daughter” triangle whose side lengths a' , $b' = na'$ and c' are also all integers. Since Theorem 2.1 can be applied again to this “daughter” triangle, and indeed applied as many times as we please, we ultimately obtain a contradiction, namely a triangle whose sides are all positive integers which are smaller than 1.

Now suppose that $\sqrt{n^2 - 1}$ is rational. This time we set $\sqrt{n^2 - 1} = a/b$ where a and b are both integers. This time our “mother” triangle will be the right angled triangle whose sides have lengths $a = b\sqrt{n^2 - 1}$, b and $c = \sqrt{a^2 + b^2} = \sqrt{(n^2 - 1)b^2 + b^2} = nb$. We can again apply Theorem 2.1 to obtain a similar smaller “daughter” triangle whose side lengths a' , b' and $c' = na'$ are also all integers. Here again, since Theorem 2.1 can be applied again to this “daughter” triangle, and indeed applied as many times as we please, we again obtain the required contradiction. \square

3. APPENDIX: THE ALGEBRAIC VERSION OF THE PAPER FOLDING PROOF OF IRRATIONALITY OF $\sqrt{2}$.

The following simple argument uses exactly the same “inductive step” as is used in the “paper folding” proof of the irrationality of $\sqrt{2}$ and also in fact in Tennenbaum’s “area of squares” proof (p. 2 of [1]). The validity of that step is justified here by much simpler (but less entertaining) algebraic reasoning instead of geometric reasoning. Let us remark that, in contrast to the “usual” algebraic proof of the irrationality of $\sqrt{2}$, in this proof we are spared the huge effort of having to make the profound realization that for each $n \in \mathbb{N}$, n^2 is even if and only if n is even.

Suppose that there exist positive integers p and q such that $\sqrt{2} = q/p$. More specifically, suppose that p is the smallest positive integer such that there exists some other integer q such that $\sqrt{2} = q/p$.

Now let q_1 and p_1 be the integers defined by

$$(2) \quad q_1 = 2p - q \quad \text{and} \quad p_1 = q - p.$$

(We remark that, in the geometric proof, if our “mother” triangle has side lengths p , p and q then the “daughter” triangle, obtained by folding it, will have side lengths p_1 , p_1 and q_1 given by (2), which is of course exactly the same as (1).)

We are going to show that $0 < p_1 < p$ and that $q_1/p_1 = \sqrt{2}$. This will contradict the assumption that p is minimal and thus prove that $\sqrt{2}$ cannot be rational.

Obviously we must have $1 < \sqrt{2} < 2$ and so $p < q < 2p$. This is exactly the same as $0 < p_1 < p$.

We have $q_1^2 = 4p^2 - 4pq + q^2$ and $p_1^2 = q^2 - 2pq + p^2$. Since $q^2 = 2p^2$ it follows that $q_1^2 = 6p^2 - 4pq$ and $p_1^2 = 3p^2 - 2pq$. Thus $q_1^2/p_1^2 = 2$ and the proof is complete. \square

More recent Acknowledgements.

7 August 2007. Thank you very much Jack Fastag for drawing my attention to websites which make much more elaborate connections between mathematics and origami than the simple ones indicated here. These sites are

<http://www.merrimack.edu/~thull/OrigamiMath.html>

where I found [2] which must surely be relevant, and also

<http://www.langorigami.com/science/science.php4>

and

<http://erikdemaine.org/>

Thank you very much Miriam Rotenberg for correcting a misprint.

REFERENCES

- [1] S. J. Miller and D. Montague, Irrationality from The Book. [arXiv:0909.4913v2](#) [[math.H0](#)]
- [2] A. Rothery, A visual proof of the irrationality of the square root of 2, Mathematics Teaching, No. 86 (March 1979), 15-16.
(I have not yet been able to obtain a copy of this article, but it seems reasonable to assume that it describes the same proof which I refer to at the beginning of this document.)

Michael Cwikel
Department of Mathematics
Technion, Israel Institute of Technology
Haifa 32000, Israel
mcwikel@math.technion.ac.il