Rotation numbers for measure-valued circle maps

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Abstract

The weak and strong topologies on the space of orbits from the unit interval to the set of probability measures are considered. A particular interest is periodic orbits of probability measures on the circle. It is shown that a real-valued rotation number can be defined in a natural way for all smooth enough orbits whose range consists of probability measures supported on the whole circle. Furthermore, this number is a continuous functional with respect to an appropriately defined strong topology. The completion of this space contains deterministic orbits as a special case, whose rotation number is an integer, coinciding with the topological degree.

1 Introduction

The objects of this study are orbits of probability measures on a compact domain $\Omega \times [0,1]$, given by $\mu = \mu_t(dx)dt$. Here μ_t is a probability (Borel) measure on Ω and $t \in [0,1]$. In particular, we are interested in the case where $\Omega \times [0,1]$ is identified with the 2-torus, i.e. Ω is the unit circle $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ and μ_t is periodic in t ($\mu_0 = \mu_1$).

Example 1 (classical orbits): Let $X(t)$ be a continuous, $1-$periodic function from $\mathbb{R}$ to $\mathbb{S}^1$. Then

$$X(t+1) = X(t) + d \quad \forall t \in \mathbb{R}$$

where $d := Deg(X)$ is an integer. Define

$$\mu_t(dx) = \delta_{X(t)} \ .$$

We can naturally identify the orbit $\mu = \mu_t dt$ with the *deterministic* orbit $X(t)$. Since X is continuous we may associate the degree $d \in \mathbb{Z}$ to this orbit.

In general, however, the concept of a degree is not a natural one for measure-valued circle maps. The periodicity of the measure $\mu_0 = \mu_1$ does not mean that *any* "particle" returns to its original position after one period. If, however, we can associate a continuous *flow* with such a map, then we may talk about the *rotation number* of this flow. This rotation number is a real number (not necessarily an integer).

Example 2 below shows that not any circle orbit can be associated with such a flow.

Example 2 (variable masses): Consider

$$\mu_t = m_1(t)\delta_{X_1(t)} + m_2(t)\delta_{X_2(t)}$$

where both $X_i(t)$, $i = 1,2$ are circle homomorphisms, and $m_i(t)$ are nonnegative, $1-$periodic functions of t satisfying $m_1(t) + m_2(t) = 1$. Suppose that the orbits of $X_i(t)$ do not intersect. Even though the latter condition implies that the degrees of X_i, $i = 1,2$ are identical, we do *not* expect this circle orbit to have neither a degree nor a rotation number, unless m_i are

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constants. An observer of such a flow notices that particles move instantly from position $X_1(t)$ to $X_2(t)$ (or vice-versa), and must assume that the velocity of these particles is infinite. Thus no flow, and no rotation number, is associated with such a measure valued circle map.

Example 3 (mixture):

$$\mu_t = \sum_1^N \beta_i \delta_{X_i(t)} \tag{1.1}$$

where X_i are deterministic orbits of corresponding degrees $d_i \in \mathbb{Z}$ and $\beta_i > 0$ are constants, $\sum_1^N \beta_i = 1$. If $d_i \neq d_j$ for some $i, j \in \{1, \dots N\}$ then the orbits X_i and X_j must intersect. An observer may assume that there is no definite velocity at the instant of intersection $X^* =: X_i(t^*) = X_j(t^*) \mod \mathbb{Z}$, as one of the particle overtakes the second. However, the observer may interpret it differently. Assume for the moment that $\beta_i = \beta_j$. Then he may claim that the two particles "exchange identity" at the moment of intersection. So, if $\beta_k = 1/N$ for $1 \leq k \leq N$ there *is* an interpretation for a *deterministic* flow transporting μ. On the other hand, the particles may not return to their initial positions after one period in time (necessarily, if $d_i - d_j$ is an odd integer). So, the rotation number of this flow, if exists, is not necessarily an integer! The same conclusion holds also if $\beta_i \neq \beta_j$. The only difference is that our observer may need now to interpret the transporting flow as a stochastic one. In any case, it is not surprising that the associated rotation number is nothing but the weighted average (see section 6)

$$r\left[\sum_1^N \beta_i \delta_{X_i} dt\right] = \sum_1^N \beta_i Deg(X_i) \ . \tag{1.2}$$

Another natural generalization of example 3 is example 2 again, but this time we assume that m_i are sequentially constants, with possible discontinuity only at intersection times of X_1 and X_2. In contrast to example 2, the "transfer of mass" between particle 1 and 2 is made only at time t when $X_1(t) = X_2(t)$, so no transfer with infinite velocity is required in this case. Again, there *is* a flow (and a rotation number) associated with the "right" interpretation in this case.

Example 4 (rigid orbits): Consider a continuous orbit of the form

$$\mu_t(dx) = g(x - X(t))dx \tag{1.3}$$

where $g(\cdot)$ is 1-periodic density and $X(t)$ is a continuous, deterministic orbit of a given degree $Deg(X) \in \mathbb{Z}$. An observer may attempt to assign the rotation number $Deg(X)$ to this orbit. However, we find out (see Proposition 6.1 below) that

$$r\left[g(x - X(t))dt\right] = Deg(X)\left(1 - \frac{1}{\int_{\mathbb{S}^1} g^{-1} dx}\right) \ . \tag{1.4}$$

In particular, the rotation number is zero if and only if the steady rotating orbit is Lebesgue measure on the circle $g \equiv 1$, since then no rotation is perceived at all.

There are several points we wish to stress:

1. The existence of a rotation number is related to the existence and the details of the associated flow, and both existence and details depend upon the interpretation. In this

paper we shall adopt the principle that the flow must be driven by a velocity field which is an $\mathbb{L}^2$ function with respect to the measure μ. In the case of mixture, for example, we shall consider only those orbits of the form (1.1) for which

$$\sum_1^N \beta_i \int_0^1 |\dot{X}_i|^2 dt < \infty \ .$$

In the case of less singular orbits μ, e.g. those orbits for which there exists $\mathbb{L}^1$ density $\mu_t = \rho(x,t)dx$, the condition for the driving velocity field to be in $\mathbb{L}^2(\mu)$ may not be sufficient to determine this field in a unique way. We shall, therefore, interpret the driving velocity as the one with *minimal* $\mathbb{L}^2(\mu)$ norm, among all consistent velocity fields.

2. There is a natural topology on the set of measure-valued orbits which makes sense for both classical orbits and mixture, on the one hand, and orbits with smooth density, on the other. This topology is induced by $C^0\left([0,1]; C^*(\mathbb{S}^1)\right)$, namely, the uniform (in t) convergence with respect to the weak, C^* topology of measures. The latter can be metrized using the Wasserstein metric W_p for any $1 \leq p < \infty$. We refer to this as the *weak* topology of measure-valued maps.

3. The rotation number cannot be a continuous function with respect to the weak topology. In particular, (1.4) implies that a sequence of steadily rotating circle orbits determined by $g_n(x - X(t))$ subjected to $\int g_n^{-1} = \infty$ admit all the same rotation number $r = Deg(X)$, while the weak limit, determined by $g(x - X(t))$, if satisfies $\int g^{-1} < \infty$, admits a different rotation number via (1.4).

4. On the other hand, the degree of a classical orbit $\mu = \delta_{x-X(t)}dt$ is continuous (and hence, a constant integer) with respect to the weak topology Thus, if we take (1.2) as a *definition* of the rotation number for mixtures (1.1), we will not be able to extend this definition in a continuous way to the weak closure of this set of mixtures. Indeed, this weak closure contains, in particular, steadily rotating measures for which the rotation number cannot be continuous with respect to the weak topology, as argued in point (3) above.

There are, basically, two different approaches to define the rotation number. The first one is to start with the set of mixtures (1.1) and define the rotation number for such mixture as (1.2). In order to extend this definition to the weak closure of mixtures, find a *strong* topology on this weak closure such that

a) The set of mixtures is dense with respect to this strong topology in the weak closure.

b) The rotation number defined by (1.2) is a continuous function with respect to the strong topology.

In this paper we shall attempt a second approach: We start from measures $\mathbf{H}^\infty$ of continuous (even smooth) and positive densities $\mu = \rho(x,t)dxdt$. For each measure $\mu \in \mathbf{H}^\infty$ there exists a *unique*, minimal driving velocity field which is smooth enough to generate a unique flow. The rotation number is defined in terms of this flow. Then, the weak closure $\mathbf{H}_2$ of $\mathbf{H}^\infty$ is defined. At the next step, a strong topology is defined on $\mathbf{H}_2$ such that $\mathbf{H}^\infty$ is dense in $\mathbf{H}_2$ with respect to this strong topology. Finally, the continuity of the rotation number with respect to the strong topology is verified. In this way, (1.2) turns out to be a theorem rather than a definition.

The relation between orbits of measures and velocity fields was discussed by several authors. Ambrosio-Gigli and Savaré [AGS] proved that if such an orbit is absolutely continuous as a mapping from $[0,1]$ to the set of measures induced with the quadratic Wasserstein metric (c.f section 2 and references cited therein), then there exists an associated velocity field transporting this orbit, which is in $\mathbb{L}^2(\mu_t)$ for a.e. $t \in [0,1]$ (this result was also quoted it [V], Exercise 8.5, p. 248). However, it is not at all evident that this field is sufficiently regular to induce a continuous flow. One of the outcomes of this paper is that, at least in the case of measure-valued circle maps, the induced field is sufficiently regular to define a rotation number in a unique way.

There is another interesting question raised by this approach. It can be easily shown that the rotation number, defined by (1.2), is continuous with respect to the weak topology on the set of mixtures (1.1), *provided* the number N of classical orbits in the mixture is uniformly bounded. Is there an analogous statement for orbits with $\mathbb{L}^1$ density $\mu = \rho(x,t)$? Point (3) above suggests the natural conjecture:

Conjecture: *The rotation number is weakly continuous on the set of measure-valued circle maps with $\mathbb{L}^1$ density $\rho = \rho(x,t)$, provided $\|\rho^{-1}\|_1$ is uniformly bounded.*

The rest of the paper is organized as follows:
In section 2 we study the weak and strong topologies on the set of measure valued orbits on a compact metric space (not restricted to circles). Starting from the set $\mathbf{H}^\infty$ of orbits subjected to smooth, positive densities, we define its weak closure $\mathbf{H}_2$ and a strong topology such that $\mathbf{H}^\infty$ is dense in $\mathbf{H}_2$ with respect to this strong topology.

In section 3 we review the notion of rotation number on circle maps.

In section 4 we discuss the set $\mathbf{H}^\infty$ within the realm of circle maps, and show that a rotation number can be defined on $\mathbf{H}^\infty$ in a unique way.

In section 5 we utilize the strong topology defined in section 2 to measure valued circle maps and prove the strong continuity of the rotation number with respect to this topology. We also prove a restricted version of the conjecture above, namely the *weak* continuity of rotation numbers under the stronger condition of bounded $\mathbb{L}^1$ norm of the *spatial* derivative of ρ^{-1}.

In section 6 we revisit examples 1 and 3, and prove the explicit expressions (1.2) and (1.4).

Finally, section 7 symmetrizes the results of this paper and suggests some ideas for further study.

2 Measure valued orbits

Let $\Omega \subset \mathbb{R}^n$ be a compact set and $P(\Omega)$ be the set of probability (Borel) measures on Ω. A measure-valued path is a functions from $[0,1]$ to $P(\Omega)$. We shall denote such an orbit by

$$\mu = \mu(dxdt) = \mu_t(dx)dt$$

with $x \in \Omega$, $t \in [0,1]$ and $\mu_t \in P(\Omega)$ for almost (Lebesgue) any t. A measure-valued path is said to admit a velocity field $\mathbf{v}(x,t) : \Omega \times [0,1] \to \mathbb{R}^n$ if the continuity equation is satisfied in the weak form:

$$\int_0^1 \int_\Omega \left[\phi_t + \mathbf{v} \cdot \nabla_x \phi\right] \mu_t(dx)dt = 0 \ , \quad \forall \phi \in C_0^\infty(\Omega \times [0,1]) \tag{2.1}$$

Given $p \geq 1$, the set $\mathbf{H}_p$ is defined as the set of all orbits μ for which a velocity field $\mathbf{v}$ exists and satisfies

$$\int_0^1 \int_\Omega |\mathbf{v}|^p \mu_t(dx)dt < \infty$$

The norm $||\mu||_p$ of $\mu \in \mathbf{H}_p$ is given by

$$||\mu||_p =: \inf_{\mathbf{v}} \left[\int_0^1 \int_\Omega |\mathbf{v}|^p \mu_t(dx)dt\right]^{1/p} \tag{2.2}$$

where the infimum is taken over all vector fields satisfying (2.1).

Recall the definition of the $p-$Wasserstein metric on probability measures:

Definition 1. *For any pair of probability measures μ_1, μ_2 on Ω, the $p-$Wasserstein metric ($p \geq 1$) is given by*

$$W_p(\mu_1, \mu_2) =: \left[\inf_\lambda \int_\Omega |x-y|^p \lambda(dxdy)\right]^{1/p}$$

where the infimum is taken on all probability measures λ on $\Omega \times \Omega$ whose marginals on Ω coincide with μ_1 and μ_2.

This metric is strongly related to the Monge-Kantorovich problem (originated by Monge [M] and relaxed by Kantorovich [K]. See [V], [R] for recent surveys).

We also recall the following proposition, whose proof can be found in [V]:

Proposition 2.1. $W_1(\mu_1, \mu_2) = \sup_{\psi \in Lip_1(\Omega)} \int_\Omega \psi(x) \left(\mu_2(dx) - \mu_1(dx)\right)$ *where $Lip_1(\Omega)$ is the set of Lipschitz functions with on Ω whose Lipschitz constant not exceeding one. If Ω is compact then W_1 is a merization of the weak topology C^* on the set of probability Borel measures on Ω.*

In fact, it follows that for compact Ω, W_p is a metrization of C^* for *any* $p \geq 1$. Note, however, that the metrics W_{p_1} and W_{p_2} are *not equivalent* unless $p_1 = p_2$. The proof of Lemma 2.1 below is given in [W].

Lemma 2.1. *If Ω is compact and $p > 1$, then any bounded set in $\mathbf{H}_p$ is uniformly $1-1/p$-Holder as a function form $[0,1]$ into $C^*(\Omega)$, where C^* is endowed with the $1-$Wasserstein metric:*

$$W_1\left(\mu_{t_2}, \mu_{t_1}\right) \leq C\left(||\mu||_p\right)|t_2 - t_1|^{1-1/p} \ .$$

By the Arzela-Ascoli Theorem and some elementary arguments we obtain [W]:

Corollary 2.1. *If $p > 1$ then for any bounded sequence $\mu_n \in \mathbf{H}_p$ there exists $\mu_\infty \in \mathbf{H}_p$ and a subsequence converging to μ_∞ uniformly on $[0,1]$ with respect to the W_1 metric. In addition, $||\cdot||_p$ is lower-semi-continuous, namely*

$$\liminf_{n\to\infty} ||\mu_n||_p \geq ||\mu_\infty||_p \ .$$

Remark 2.1: In the case $\mu = \delta_{x(t)}dt$ with $x(t) \in \Omega$ for $t \in [0,1]$ then $||\mu||_p$ is reduced to the $\mathbb{L}^p$ norm of $\dot{x} := dx/dt$.

Remark 2.2: Consider the path $\mu = \sum_1^k m_i(t)\delta_{x_i(t)}$, where $m_i(t)$ are smooth, positive functions satisfying $\sum_1^k m_i(t) = 1$ and x_i smooth orbits from $[0,1]$ to Ω. Then $\mu \notin \mathbf{H}_p$ for any $p \geq 1$, unless m_i are constants. However, there exists a bounded sequence $\{\mu_n\} \subset \mathbf{H}_1$ which converges uniformly to μ in the W_1 metric. This example demonstrates the necessity of $p > 1$ in Corollary 2.1.

Definition 2. *A sequence $\{\mu_n\} \subset \mathbf{H}_p$ is said to be weakly converging to $\mu \in \mathbf{H}_p$ whenever it is uniformly bounded in the $\mathbf{H}_p$ norm and converges to μ in $C\left([0,1]; C^*(\mathbb{S}^1)\right)$ where C^* is equipped with the W_2 metric.*

Since the metrics W_1 and W_2 metrize the same topology $C^*(\Omega)$ for compact Ω, we obtain from Corollary 2.1:

Lemma 2.2. *The set $\mathbf{H}_2$ is locally sequentially compact with respect to the weak topology.*

The example in Remark 2.2 also demonstrates that not every measure path admits a velocity field. Even if $\mu \in \mathbf{H}_p$, the associated vector field may not be smooth enough to generate a flow. However, by compactness we easily obtain ([W]):

Lemma 2.3. *For any $\mu \in \mathbf{H}_p$, $p > 1$ there exists a vector field, defined μ a.e, which realizes the infimum in (2.2).*

From now on, we shall consider only the case $p = 2$.

Definition 3. $\mathbf{H}^\infty$ *is the set of all paths $\mu = \rho(x,t)dxdt$, $\int_\Omega \rho(x,t)dx = 1$ for any $t \in [0,1]$, such that ρ is smooth and strictly positive on $\overline{\Omega} \times [0,1]$.*

Lemma 2.4 below is a special case of known results. See [BB, BBG] for informal introduction and [O] for closely related results on the Riemannian structure of Monge-Kantorovich flows. The extension for non-smooth case is given in [AGS]. Chapter 8 of [V] contains an illuminating review of these recent results.

Lemma 2.4. *Assume that $\Omega \subset \mathbb{R}^n$ is compact with smooth boundary and $\mu \in \mathbf{H}^\infty$. Then $\mu \in \mathbf{H}_2$. Moreover, there exists a unique, optimal vector field which minimize (2.2) in the $p = 2$ case. This optimal vector field is a gradient, given by the unique (up to a constant) solution of the elliptic PDE:*

$$\nabla_x \cdot (\rho(x,t)\nabla_x \phi) = -\frac{\partial \rho}{\partial t} \quad ; \quad \nabla_x \phi \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \quad \forall t \in [0,1]$$

where $\boldsymbol{n}$ is the normal to a point on $\partial\Omega$.

The Regularization Theorem below is the $p = 2$ case of known results. See [Am] in the case $p = 1$ and [AGS] for $p > 1$. See also [W].

Theorem 2.1. Regularization Theorem*: The space $\mathbf{H}_2$ is the weak closure of $\mathbf{H}^\infty$. That is, for any $\mu \in \mathbf{H}_2$ there exists a sequence $\{\mu_n\} \subset \mathbf{H}^\infty$ where*

1. *$\{\mu_n\}$ is uniformly bounded in the $\mathbf{H}_2$ metric.*
2. *μ_n converges to μ uniformly w.r to $[0,1]$ in the W_1 metric.*

Moreover, for any $\mu \in \mathbf{H}_2$ there exists such a sequence for which

$$\lim_{n\to\infty} ||\mu_n||_2 = ||\mu||_2 \ . \tag{2.3}$$

The next object is to define an inner-product on $\mathbf{H}^\infty$. This will enable us to define a *strong* topology on $\mathbf{H}_2$, alongside the weak one given by Definition 2. For this, we introduce an additional characterization of the Wasserstein metric W_2:

Definition 4. *Given a pair of Borel probability measures μ_1, μ_2 on a common probability space Ω, the push-forward of μ_1 to μ_2 is a mapping $T : \Omega \to \Omega$ satisfying*

$$\mu_1\left(T^{-1}\mathcal{A}\right) = \mu_2\left(\mathcal{A}\right) \quad \forall \text{Borel sets } \mathcal{A} \subset \Omega \quad . \tag{2.4}$$

We shall often use the common notation $T_\# \mu_1 = \mu_2$.

Note that an equivalent definition for $T_\# \mu_1 = \mu_2$ is

$$\int_\Omega \phi(T(x))\mu_1(dx) = \int_\Omega \phi(x)\mu_2(dx) \tag{2.5}$$

for any continuous ϕ.
From [Am] we know that if the measure μ_1 contains no atom then the Wasserstein-2 metric W_2, given in Definition 1, is also given as

$$W_2(\mu_1,\mu_2) = \sqrt{\inf_T \int_\Omega |x - T(x)|^2 \mu_1(dx)} \tag{2.6}$$

where the infimum above is in the class of all maps satisfying (2.4). If, however, μ_1 contains an atom then the set of mappings T verifying Definition 4 can be empty.

We recall the fundamental result of Brenier ([B], see also [G-M]): If μ_1 is absolutely continuous with respect to Lebesgue measure, then there is a unique optimal Borel map T satisfying (2.4) and realizing the minimum in (2.6). Moreover, this optimal transport T is the gradient of a convex function Ψ.

Choose now a reference probability measure $\gamma(dx)$ on Ω, absolutely continuous with respect to Lebesgue (for simplicity, the normalized Lebesgue measure will do, since Ω is compact). Given $\mu = \mu_t dt \in \mathbf{H}_2$ define $T^{(\mu,t)} : \Omega \to \Omega$ and

$$T^{(\mu)}(x,t) := \left(T^{(\mu,t)}(x), t\right) \quad \forall t \in [0,1] \ .$$

such that for $t \in [0,1]$, $T^{(\mu,t)}$ is the W_2-optimal map transporting γ to μ_t, that is:

$$T^{(\mu,t)}_{\#}\gamma = \mu_t \quad \text{and} \quad W_2\left(\mu_t, \gamma\right) = \sqrt{\int_\Omega \left|x - T^{(\mu,t)}(x)\right|^2 \gamma(dx)} \ .$$

Given $\mu = \rho(x,t)dxdt \in \mathbf{H}^\infty$. By Lemma 2.4 we may associate the unique, smooth optimal velocity field $\boldsymbol{v}_\mu = \nabla_x \phi_\rho(x,t)$ with this μ. Let

$$\boldsymbol{w}_\mu(x,t) := \boldsymbol{v}_\mu\left(T^{(\mu,t)}(x), t\right) \ . \tag{2.7}$$

Then

$$\int_0^1 \int_\Omega |\boldsymbol{v}_\mu|^2 \mu_t(dx)dt = \int_0^1 \int_\Omega |\boldsymbol{w}_\mu|^2 \gamma(dx)dt = ||\mu||_2^2 \ . \tag{2.8}$$

Definition 5. *Let $\mu^{(1)}, \mu^{(2)} \in \mathbf{H}^\infty$, $\boldsymbol{v}_1, \boldsymbol{v}_2$ the associated velocity fields and $\boldsymbol{w}_1, \boldsymbol{w}_2$ the $\gamma(dx)dt$ measurable functions associated with $\boldsymbol{v}_1, \boldsymbol{v}_2$ via (2.7). The induced inner product*

$$\left\langle \mu^{(1)}, \mu^{(2)} \right\rangle_\gamma := \int_0^1 \int_\Omega \boldsymbol{w}_1(x,t) \cdot \boldsymbol{w}_2(x,t) \gamma(dx)dt \tag{2.9}$$

defines a metric on $\mathbf{H}^\infty$ via:

$$D_\gamma\left(\mu^{(1)}, \mu^{(2)}\right) := \sqrt{\int_0^1 \int_\Omega |\boldsymbol{w}_1(x,t) - \boldsymbol{w}_2(x,t)|^2 \gamma(dx)dt} + \sup_{t\in[0,1]} W_2\left(\mu_t^{(1)}, \mu_t^{(2)}\right) \ . \tag{2.10}$$

Let $\mathbf{H}_2^c$ be the closure of $\mathbf{H}^\infty$ with respect to the metric D_γ.

Theorem 2.2. $\mathbf{H}_2^c \equiv \mathbf{H}_2$.

To prove Theorem 2.2 we need the following:

Lemma 2.5. *Let $\left\{\mu^{(n)}\right\}_1^\infty \subset \mathbf{H}^\infty$ converges weakly to $\mu \in \mathbf{H}_2$. Let $\boldsymbol{v}_n \in \mathbb{L}^2\left(\mu^{(n)}\right)$ be the transport velocity field corresponding to $\mu^{(n)}$ (defined uniquely by Lemma 2.4), and $\boldsymbol{w}_n \in \mathbb{L}^2\left(\gamma(dx)dt\right)$ the corresponding vector field via (2.7). Then there exists a subsequence for which $\boldsymbol{w}_n$ converges $\mathbb{L}^2-$weakly to some $\boldsymbol{w} \in \mathbb{L}^2\left(\gamma(dx)dt\right)$ and*

$$\int_0^1 \int_\Omega |\boldsymbol{w}|^2 \gamma(dx)dt \geq ||\mu||_2^2 \ .$$

Proof. of Theorem 2.2:
Given $\mu \in \mathbf{H}_2$, by Theorem 2.1 there exists a sequence $\left\{\mu^{(n)}\right\}_1^\infty \subset \mathbf{H}^\infty$ which converges weakly to μ and, moreover, satisfies (2.3). By definition of the weak convergence of $\mu^{(n)}$ it is a Cauchy sequence with respect to the metric $C([0,1], C^*)$ where C^* is equipped with the W_2 metric. Hence the second part of the metric D_γ is Cauchy. We have to show that there exists a subsequence for which $\boldsymbol{w}_n$ is also Cauchy in the strong $\mathbb{L}^2(\gamma dt)$ metric. By Lemma 2.5 we can extract such a subsequence for which $\boldsymbol{w}_n$ converges $\mathbb{L}^2-$ weakly to $\boldsymbol{w} \in \mathbb{L}^2(\gamma dt)$ and, moreover, $||\boldsymbol{w}||_2 \geq ||\mu||_2^2$. In addition, (2.3) and (2.8) imply that

$$||\mu||_2^2 = \lim_{n\to\infty} ||\mu^{(n)}||_2^2 = \lim_{n\to\infty} ||\boldsymbol{w}_n||_2^2 \ . \tag{2.11}$$

It follows that $||\boldsymbol{w}||_2^2 \geq \lim_{n\to\infty} ||\boldsymbol{w}_n||_2^2$. Since $\boldsymbol{w}$ is a weak limit of $\boldsymbol{w}_n$ in $\mathbb{L}^2(\gamma dt)$ it follows that in fact, $||\boldsymbol{w}||_2^2 = \lim_{n\to\infty} ||\boldsymbol{w}_n||_2^2$ and that the convergence is strong. $\square$

Proof. of Lemma 2.5:
Since $||\mu^{(n)}||_2$ are uniformly bounded by assumption, then $\boldsymbol{w}_n$ are uniformly bounded on $\Omega \times [0,1]$ with respect to the $\mathbb{L}^2(\gamma dt)$ norm. In particular, there exists a subsequence of $\boldsymbol{w}_n$ which converge to $\boldsymbol{w} \in \mathbb{L}^2(\gamma dt)$ in the weak topology of $\mathbb{L}^2$.

Set $\boldsymbol{m}_n := \boldsymbol{v}_n \mu^{(n)}$. Since $\int_0^1 \int_\Omega |\boldsymbol{m}_n| \leq \left(\int_0^1 \int_\Omega |\boldsymbol{v}_n|^2 \mu_t^{(n)}(dx)dt\right)^{1/2}$ is bounded by assumption, there exists a subsequence for which $\boldsymbol{m}_n$ converges weak$-*$ to a vector valued Borel measure $\boldsymbol{m}$. We claim that $|\boldsymbol{m}|$ is absolutely continuous with respect to μ. In fact, for each continuous $0 \leq \psi \leq 1$ on $\Omega \times [0,1]$ we have:

$$\int_0^1 \int_\Omega \psi d|\boldsymbol{m}_n| \leq \int_0^1 \int_\Omega \sqrt{\psi} d|\boldsymbol{m}_n| \leq \left(\int_0^1 \int_\Omega \psi \mu_t^{(n)}(dx)dt\right)^{1/2} ||\mu^{(n)}||_2^2$$

which implies that, in the limit, $\int_0^1 \int_\Omega \psi d|\boldsymbol{m}| \leq C \left(\int_0^1 \int_\Omega \psi \mu_t(dx)dt\right)^{1/2}$ as well. Let $\boldsymbol{v}$ be the Radon-Nikodym derivative of $\boldsymbol{m}$ with respect to μ. In addition, the weak convergence of both $\mu^{(n)}$ and $\boldsymbol{m}_n$ implies that $\partial_t \mu_t + \nabla_x \cdot \boldsymbol{m} = 0$ in the sense of distribution, so $\boldsymbol{v}$ is, indeed, a transporting velocity field of $\mu \in \mathbf{H}_2$ (possibly not a unique or optimal one), and

$$||\mu||_2^2 \leq \int_0^1 \int_\Omega |\boldsymbol{v}|^2 \mu_t(dx)dt \ . \tag{2.12}$$

Define now $T^{(t)}$ to be the optimal W_2- map transporting γ to μ_t. Likewise, $T_n^{(t)}$ the optimal W_2- map transporting γ to $\mu_t^{(n)}$. For any test function $\phi \in C^\infty(\Omega \times [0,1])$ we have:

$$\int_0^1 \int_\Omega \phi(x,t) \boldsymbol{v}(x,t) \mu_t(dx)dt = \int_0^1 \int_\Omega \phi\left(T^{(t)}(x), t\right) \boldsymbol{v}\left(T^{(t)}(x), t\right) \gamma(dx)dt \ . \tag{2.13}$$

The l.h.s of (2.13) is the limit of

$$\int_0^1 \int_\Omega \phi(x,t) \boldsymbol{v}_n(x,t) \mu_t^{(n)}(dx)dt = \int_0^1 \int_\Omega \phi(T_n^{(t)}(x), t) \boldsymbol{w}_n(x,t) \gamma(dx)dt \ . \tag{2.14}$$

We claim now that

$$\lim_{n\to\infty}\int_0^1\int_\Omega \phi(T_n^{(t)}(x),t)\boldsymbol{w}_n(x,t)\gamma(dx)dt = \int_0^1\int_\Omega \phi(T^{(t)}(x),t)\boldsymbol{w}(x,t)\gamma(dx)dt \ . \tag{2.15}$$

Granted (2.15) we obtain from (2.13) that

$$\int_0^1\int_\Omega \phi\left(T^{(t)}(x),t\right)\boldsymbol{v}\left(T^{(t)}(x),t\right)\gamma(dx)dt = \int_0^1\int_\Omega \phi(T^{(t)}(x),t)\boldsymbol{w}(x,t)\gamma(dx)dt \tag{2.16}$$

holds for any test function ϕ.

Let now $\mathbf{W}_T$ be the closed subspace of $\mathbb{L}^2(\gamma dt)$ defined by

$$\mathbf{W}_T = \overline{\left\{\phi_T := \phi(T^{(t)}(x),t) \quad ; \quad \phi \in C(\Omega\times[0,1])\right\}}_{\mathbb{L}^2(\gamma dt)} \ .$$

Then, (2.16) implies that

$$\boldsymbol{w} = \boldsymbol{v}\circ T + \boldsymbol{q}$$

where $\boldsymbol{q}\in \mathbf{W}_T^\perp$. Hence

$$||\boldsymbol{w}||^2_{2,\gamma dt} = ||\boldsymbol{v}\circ T||^2_{2,\gamma dt} + ||\boldsymbol{q}||^2_{2,\gamma dt} \geq ||\boldsymbol{v}\circ T||^2_{2,\gamma dt} = \int_0^1\int_\Omega |\boldsymbol{v}|^2\mu_t(dx)dt$$

which completes the proof via (2.12).

To prove (2.15) it is enough to show that the sequence $\phi_n := \phi(T_n^{(t)}(x),t)$ converges $\mathbb{L}^2(\gamma dt)-$*strongly* to $\phi_\infty := \phi(T^{(t)}(x),t)$. By the assumed smoothness of ϕ, the compactness of $\Omega\times[0,1]$ and the dominated convergence theorem it is enough to show that

$$\lim_{n\to\infty}\int_0^1\int_\Omega \left|T_n^{(t)}(x) - T^{(t)}(x)\right|^2\gamma(dx) = 0 \tag{2.17}$$

holds for any $t\in[0,1]$.

The proof of (2.17) follows by an adaptation of the proof of Lemma 5.1 in [EGH] (see also Theorem 1.1.7 in [R]). $\square$

3 Measure valued Circle maps

Here we are interested in *periodic* orbits of probability measures on the circle.

Definition 6. *The set* $\mathbf{H}_C$ *consists of all orbits* $\mu = \mu_t dt$ *where, for any* $t\in[0,1]$, μ_t *is a Borel probability measure on the circle* $\mathbb{S}^1$ *and* $\mu_0 = \mu_1$.

Let us consider now a *smooth* velocity function $v = v(x,t)$ on the 2-torus $\mathbb{S}^1\times\mathbb{S}^1$. We may, of course, identify it with a 1×1 periodic function on $\mathbb{R}^2$. This velocity function generates a flow $Y:\mathbb{R}^2\to\mathbb{R}$, via

$$\frac{\partial Y}{\partial t} = v(Y,t) \quad ; \quad Y(x,0) = x$$

The time one mapping of this flow, denoted by $H(x) =: Y(x,1)$, is a circle homomorphism. To see this, one has to observe that $Y(x+1,t) = Y(x,t)+1$, hence $H(x+1) = H(x)+1$.

We associate a rotation number for a circle map, defined by

$$\tilde{r} = \lim_{n\to\infty} \frac{H^{(n)}(x)}{n}$$

where $H^{(1)}(x) = H(x)$, $H^{(n+1)}(x) = H^{(n)}(H(x))$. It is known that $\tilde{r} \in [0,1)$ is independent of the choice of x. In our case H is originated from a flow Y so we shall define the rotation number of the *flow* as

$$r = \lim_{t\to\infty} \frac{Y(x,t)}{t} \ .$$

According to this definition, r can be any real number. Evidently, $\tilde{r}$ is the fractional part of r.

Can we associate $\mathbf{H}_C$ with such a flow Y? Let μ_0 be any Borel probability measure on the circle. We may define (see Definition 4)

$$\mu_t = Y_{\#}(\cdot, t)\mu_0 \ .$$

However, $\mu = \mu_t dt \notin \mathbf{H}_C$, in general, unless $\mu_1 = \mu_0$.

For a comprehensive text on circle maps see [Ar]. Let us recall the following fact concerning circle maps: For *any* circle map H there is at least one *invariant measure* ζ, namely $H_{\#}(\zeta) = \zeta$.

Lemma 3.1. *An orbit $\mu_t = Y_{\#}(\cdot, t)\mu_0$ is in $\mathbf{H}_C$ if and only if μ_0 is an invariant measure of the associated homomorphism $H = Y(\cdot, 1)$.*

So, how many choices do we have to determine a $\mu \in \mathbf{H}_C$, once a flow Y is given? It is certainly the same as the number of invariant measures associated with the corresponding circle homomorphism. Let us consider all possible cases:

1. If r is irrational, then there is a unique invariant measure. Moreover, this measure is supported on the whole circle.

2. If the rotation number is rational $r = p/q$, then the homomorphism $H^{(q)}$ must have fixed points. Each such fixed point x_0 represents a $q-$periodic orbit of H, given by $x_i = H^{(i)}(x_0)$, $i = 0, \dots q-1$. For each such periodic orbit, the measure $\zeta = \frac{1}{q}\sum_{i=0}^{q-1} \delta_{x_i}$ is an invariant measure.

3. As a special case of (2), let us consider the case where H is the *identity* map. Then *any* measure ζ is invariant. In this case, the rotation number r is not only a rational but an integer, given by the degree of the orbits.

A less familiar fact, whose proof is rather easy, is the following:

Proposition 3.1. *If an invariant measure ζ of a circle map H is supported on the whole circle, then either*

(i) r is irrational, or

ii) $r = p/q$, and $H^{(q)}$ is the identity map.

Example: The canonical example of a circle map is the rigid rotation given by

$$R_r(x) = x + \hat{r} \mod 1$$

where $\hat{r} \in [0,1)$. The associated rotation number is evidently $\hat{r}$. The invariant measure is the uniform Lebesgue measure on $\mathbb{S}^1$. If r is irrational then it is the *only* invariant measure associated with R_r. If $r = p/q$ then any measure $\zeta = \rho(x)dx$ is an invariant measure, provided ρ is $1/q-$periodic, namely $\rho(x + 1/q) = \rho(x)$.

Notation: The *cumulation function* F_ζ associated with a probability measure ζ on $\mathbb{S}^1$ is a monotone non-decreasing, right continuous function on the line such that, for any $0 < x < y \leq 1$:

$$F_\zeta(y) - F_\zeta(x) = \zeta(x, y] \ , \tag{3.1}$$

and $F_\zeta(x+1) = F_\zeta(x)+1$. We shall also use the convention $F_\zeta(0_-) = 0$ which, together with (3.1), determines F_ζ uniquely.

We may now introduce the function $F = F(x,t)$ as a representation of $\mu \in \mathbf{H}_C$. The definition of $\mathbf{H}_C^\infty$ below is consistent with Definition 3 of section 2:

Definition 7. *The set* $\mathbf{H}_C^\infty$ *is composed of all orbits* μ *represented by* $F = F(x,t)$ *such that:*

i) $F(x+1,t) = F(x,t) + 1$ *and* $F(x,t) = F(x,t+1)$ *for any* $x, t \in \mathbb{R}$ *(namely,* $\mu \in \mathbf{H}_C$*).*

ii) F *is a smooth function of* x, t*, and* $F_x > 0$ *for any* $x, t \in \mathbb{R}$*.*

Let now F be a representation of $\mu \in \mathbf{H}_C^\infty$. For any arbitrary periodic function $\lambda = \lambda(t)$, let

$$v(x,t) = \frac{\lambda(t) - F_t}{F_x} \tag{3.2}$$

that is, the velocity field satisfies

$$F_t + vF_x = \lambda(t) \ .$$

Recall that $\mu_t(dx) = \rho(x,t)dx$ where $\rho = F_x(x,t)$, so the velocity field v transports the orbit μ, that is the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho v\right)}{\partial x} = 0 \tag{3.3}$$

is satisfied *pointwise*. Let $Y = Y(x,t)$ be the flow induced by v:

$$\frac{\partial Y}{\partial t} = v\left(Y(x,t), t\right) \quad ; \quad Y(x,0) = x \ \ \forall x \in \mathbb{R} \ . \tag{3.4}$$

From Definition 7 we obtain the following characterization of $\mathbf{H}_C^\infty$:

Lemma 3.2. *Let* $\mu \in \mathbf{H}_C^\infty$*,* Y *the corresponding flow* (3.4) *and* $H = Y(,1)$ *then*

i) μ_0 *is an invariant measure of* $H = Y(,1)$*.*

ii) μ_0 *has a smooth, (strictly) positive density* $\rho_0(x)dx$ *on* $\mathbb{S}^1$*.*

4 Rotation numbers on $\mathbf{H}_C^\infty$

In general, a vector field may not be smooth enough to generate a flow. If we restrict ourselves to $\mu \in \mathbf{H}_C^\infty$, then a smooth vector field exists by definition. The identity (3.2) yields all vector fields associated with $\mu \in \mathbf{H}_C^\infty$ in terms of the cumulation function F associated with μ and arbitrary, 1−periodic functions λ. Hence, the norm $||\mu||_2$ as given by (2.2) is

$$||\mu||_2 = \sqrt{\min_\lambda \int_0^1 \int_0^1 \left[\frac{\lambda - F_t}{F_x}\right]^2 F_x dx dt} = \sqrt{\min_\lambda \int_0^1 \int_0^1 \frac{(\lambda - F_t)^2}{F_x} dx dt} \ . \tag{4.1}$$

where the minimum is taken over all 1−periodic functions $\lambda = \lambda(t)$.

Definition 8. *The set $\mathbf{H}_{C,2}$ is the weak closure of $\mathbf{H}_C^\infty$ with respect to the norm $||\cdot||_2$, where weak convergence is understood as in Definition 2.*

Now, the optimal velocity field v is the one minimizing the action in (4.1). This condition determines λ via

$$\lambda(t) = \frac{\int_0^1 \frac{F_t(x,t)}{F_x(x,t)} dx}{\int_0^1 \frac{1}{F_x(x,t)} dx} \ .$$

Recalling $F_x = \rho$ and denoting the w average of a function f, where $w \geq 0$ on $[0,1]$ as

$$\langle f \rangle_w =: \frac{\int_0^1 w(x) f(x) dx}{\int_0^1 w(x) dx}$$

we may rewrite $\lambda(t)$ as

$$\lambda(t) = \langle F_t(\cdot, t)\rangle_{\rho^{-1}(\cdot,t)} \ . \tag{4.2}$$

The optimal velocity field is given now, via (3.2), by

$$\overline{\boldsymbol{v}}(x,t) = \rho^{-1}(x,t) \left[\langle F_t(\cdot,t)\rangle_{\rho^{-1}(\cdot,t)} - F_t(x,t)\right] \ .$$

In particular, it follows that $\int_0^1 \overline{\boldsymbol{v}}(x,t) dx = 0$ for any t, so

Lemma 4.1. *If $\mu = \rho(x,t) dx dt \in \mathbf{H}_C^\infty$, then the optimal velocity field is given by a potential defined on the circle $\mathbb{S}^1$, namely*

$$\overline{\boldsymbol{v}}(x,t) = \phi_x(x,t) = \rho^{-1}(x,t) \left[\langle F_t(\cdot,t)\rangle_{\rho^{-1}(\cdot,t)} - F_t(x,t)\right] \tag{4.3}$$

where ϕ is 1-periodic in x.

Note that ϕ_x is 1-periodic in t by definition and its integral with respect to x is zero on a period. Hence ϕ can be defined as 1-periodic in both x and t, i.e a function on the 2−torus.

We shall now define the rotation number of $\mu \in \mathbf{H}_C^\infty$ as follows

Definition 9. *The rotation number of $\mu \in \mathbf{H}_C^\infty$ is the rotation number of the flow due to the optimal velocity field ϕ_x associated with μ.*

Using Lemma 4.1 and (4.3) we can give an explicit expression for $||\mu||_2$. Moreover, the rotation r can also be computed explicitly:

Lemma 4.2. *If $\mu \in \mathbf{H}_C^\infty$, the rotation number is given by:*

$$r = \int_0^1 \langle F_t(\cdot, t)\rangle_{\rho^{-1}(\cdot,t)} dt \tag{4.4}$$

and

$$||\mu||_2^2 = \int_0^1 \left[\langle F_t^2\rangle_{\rho^{-1}} - \langle F_t\rangle_{\rho^{-1}}^2\right] \left(\int_0^1 \rho^{-1}(x,t)dx\right) dt \tag{4.5}$$

Proof. Let $x(t)$ be one of the orbits of the associated flow. Then

$$\frac{d}{dt}F(x(t),t) = F_x(x(t),t)\dot{x} + F_t(x(t),t)$$

But $\dot{x} = v(x,t)$ so, by (4.3):

$$\frac{d}{dt}F(x(t),t) = \langle F_t(\cdot,t)\rangle_{\rho^{-1}(\cdot,t)}$$

Since $F(x,t) - x$ is a periodic function (in both x and t), it follows that $|F(x(t),t) - x(t)|$ is uniformly bounded, hence

$$r = \lim_{t\to\infty} \frac{x(t)}{t} = \lim_{t\to\infty} \frac{F(x(t),t)}{t} = \int_0^1 \langle F_t(\cdot,t)\rangle_{\rho^{-1}(\cdot,t)} dt$$

where the last equality follows from the $1-$ periodicity of $\langle F_t(\cdot,t)\rangle_{\rho^{-1}(\cdot,t)}$.

Next, we use (4.3) and the definition of $\langle\cdot\rangle_{\rho^{-1}}$ to obtain (4.5) via

$$||\mu||_2^2 = \int_0^1 \int_0^1 \rho(x,t)|\overline{v}|^2 dxdt = \int_0^1 \int_0^1 \frac{\left(\langle F_t\rangle_{\rho^{-1}} - F_t\right)^2}{\rho(x,t)} dxdt \ .$$

$\square$

5 Rotation numbers on the extended circle maps

5.1 Strong continuity

Let $\mathbf{H}_{C,2}$ the set of orbits obtained by the weak closure of $\mathbf{H}_C^\infty$. Our first object is to show that the rotation number r can be defined on $\mathbf{H}_{C,2}$ in a unique way. For this, we shall present a different representation for $||\cdot||_2$ and r on $\mathbf{H}_C^\infty$ as follows:

The cumulation function $F = F(x,t)$ corresponding to μ is defined only up to an additive periodic function of t. Indeed, condition (i) of Definition 7 is still satisfied if we replace $F(x,t)$ by $F(x,t) + \xi(t)$ where ξ is a *periodic* function $\xi(t+1) = \xi(t)$. In particular, $F(x,t)$ and $F(x,t) + \xi(t)$ correspond to the same orbit μ.

Let $X = X(F,t)$ the inverse of F for given t. The gauge freedom in the definition of F induces the corresponding freedom in the definition of its inverse, namely $X(F+\xi(t),t)$ and $X(F,t)$ represent the same orbit μ. Note that X satisfies the conditions:

$$X(F+1,t) = X(F,t)+1 \;\; ; \;\; X(F,t+1) = X(F) \; . \tag{5.1}$$

We shall take advantage of the gauge freedom to define:

Definition 10. *Given $\mu \in \mathbf{H}_{C,2}$, the* **standard gauge** *for X is the one defined by*

$$\int_0^1 X(F,t)dF = \frac{1}{2} \quad \forall t \in \mathbb{R} \; . \tag{5.2}$$

Remark: Since X is monotone in F for any t there exists a unique standard gauge for any such μ.
Remark: The W_p- distance between a pair of Borel measures μ_1 and μ_2 on $\mathbb{S}^1$ is obtained via

$$\left[\inf_\omega \int_0^1 |X_1(F+\omega) - X_2(F)|^p \, dF \right]^{1/p} \quad ,$$

where X_i are the inverse of the cumulative functions corresponding to μ_i. In particular, for any pair of orbits $\mu^{(1)}, \mu^{(2)}$, the second part of the metric D_γ, as defined in Definition 5, is given by

$$\sup_t \sqrt{ \inf_\omega \int_0^1 |X_1(F+\omega,t) - X_2(F,t)|^2 \, dF } \tag{5.3}$$

where, again, $X_i(F,t)$ is the inverse cumulation function of $\mu_t^{(i)}$. Note that the distance given by (5.3) is independent of the chosen gauge.

Lemma 5.1. *Let l be the Lebesgue measure on $\mathbb{S}^1$, $\mu \in \mathbf{H}^\infty_C$ and $X = X(F,t)$ be the corresponding inverse of the cumulation function associated with μ. Then, for any fixed $t \in \mathbb{R}$, the function $X = X(\cdot, t) : \mathbb{S}^1 \to \mathbb{S}^1$ is the optimal W_2-map transporting l to μ_t iff X is taken with the standard gauge (5.2).*

Proof. Let $\psi \in C^\infty(\mathbb{R})$ be a $1-$periodic function. First, note that by an elementary change of variables and (5.1):

$$\int_0^1 \phi(x)\mu_t(dx) = \int_0^1 \phi(x) F_x(x,t)dx = \int_0^1 \phi(X(F+\omega,t))dF$$

holds for any $\omega \in \mathbb{R}$. Hence $X(t,\cdot)_\# l = \mu_t$ holds for *any* gauge.

So, let us fix a special gauge for which $X(0,t) = 0$ for both the Lebesgue measure l and μ_t for all $t \in [0,1]$. In this gauge, the Lebesgue measure l is identified with $X_l(F) = F$. Let F_0 and X_0 be the cumulation function of μ and its inverse in this special gauge, namely $F_0(0,t) = X_0(0,t) = 0$ for any t. The optimal W_2-map is given by minimizing

$$\Xi(\omega,t) := \int_0^1 |X_0(F+\omega,t) - F|^2 \, dF$$

for any t. The condition $\Xi'_{\omega}(\omega, t) = 0$ implies

$$0 = \int_0^1 (X_0(F+\omega, t) - F)\, X'_{0,F}(F+\omega, t) dF = \int_{\omega}^{1+\omega} (X_0(F, t) - F + \omega)\, X'_{0,F}(F+, t) dF$$

$$= \int_{X_0(\omega)}^{X_0(\omega)+1} [x - F_0(x, t) + \omega]\, dx = \frac{1}{2} + \omega + X_0(\omega) - \int_{X_0(\omega)}^{X_0(\omega)+1} F_0(x, t) dx \ .$$

A direct computation, based on Definition 7-(i), yields

$$\int_{X_0(\omega)}^{X_0(\omega)+1} F_0(x, t) dx = \int_0^1 F_0(x, t) dx + X_0(\omega) \ ,$$

which, together with the former line, yields

$$\int_0^1 F_0(x, t) dx = \frac{1}{2} + \omega \ .$$

Next, note that, since X_0 and F_0 are inverse pair in the special gauge, the equality $\int_0^1 F_0(x, t) dx + \int_0^1 X_0(F, t) dF = 1$ holds. Hence

$$\int_0^1 X_0(F, t) dF = \frac{1}{2} - \omega \ .$$

Finally, the gauge $X_0(F, t) \to X(F, t) = X_0(F+\omega, t)$ yields, via (5.1):

$$\int_0^1 X(F, t) dF = \int_0^1 X_0(F, t) dF + \omega \Longrightarrow \int_0^1 X(F, t) dF = \frac{1}{2} \ .$$

$\square$

Definition 11. *Let $a = a(t)$ a periodic function on $\mathbb{R}$ and $A = A(t)$ its primitive. For any function $Y(F, t)$ which is periodic in both F and t, the shifted function $S_A Y$ is defined as $S_A Y(F, t) := Y(F + A(t), t)$.*

Next, since $X(F(x, t), t) \equiv x$ so

$$\frac{dX}{dt}(F(x, t), t) = X_F F_t + X_t = 0 \tag{5.4}$$

and $X_F = F_x^{-1}\big|_{x=X}$. Then, using (5.4) and Definition 11

$$S_{-A}\left(\frac{\partial}{\partial t} S_A X\right) = \left.\frac{a(t) - F_t}{F_x}\right|_{x=X(F,t)} = \boldsymbol{v}\left(X(F, t), t\right) \ . \tag{5.5}$$

From (3.2) we observe that $\boldsymbol{v}(x, t)$ is a transporting velocity associated with the orbit $\mu = F_x dx dt$. In particular, if $\Lambda(t)$ is the primitive function of $\lambda(t)$ defined as in (4.2), and

$$\overline{X}(F, t) := S_{\Lambda} X(F, t) \ ,$$

Then

$$S_{-\Lambda} \overline{X}_t = \overline{\boldsymbol{v}}\left(X(F, t), t\right) \ , \tag{5.6}$$

where $\overline{\boldsymbol{v}}$ is the *optimal* velocity field transporting μ (see (4.2), (4.3)).

Lemma 5.2. *Let $\mu \in \mathbf{H}_C^\infty$ and X the inverse of the corresponding cumulation function. Then*

$$||\mu||_2^2 = \int_0^1 \int_0^1 \left|\overline{X}_t\right|^2 dF dt = \inf_A \int_0^1 \int_0^1 |(S_A X)_t|^2 \, dF dt \ . \tag{5.7}$$

Proof. From (2.2) with $p = 2$, (5.5) and (5.6) it follows that

$$||\mu||_2^2 = \int_0^1 \int_0^1 \left|S_{-\Lambda}\overline{X}_t\right|^2 dF dt = \inf_A \int_0^1 \int_0^1 \left|S_{-A}\left(S_A X\right)_t\right|^2 dF dt \ .$$

The shift operator S_A leaves invariant the integral over one period of F, so (5.7) follows. $\square$

Remarks:

i) The function $\Lambda(t)$ depends, of course, on the gauge. A different gauge corresponds to a change of $\Lambda \to \Lambda + \xi$ where ξ is a periodic function.

ii) Even with a fixed gauge, the function Λ, defined by (5.7), is determined only up to a constant. We make the convention that $\int_0^1 \Lambda(t)dt = 0$ in order to determine it completely.

The rotation number r can be written as

$$r = \Lambda(1) - \Lambda(0) = \int_0^1 \int_0^1 \lambda(t) dx dt = \int_0^1 \int_0^1 \frac{\lambda(t)}{F_x(X(F,t),t)} dF dt \ . \tag{5.8}$$

while

$$\int_0^1 \int_0^1 F_t(x,t) dx dt = \int_0^1 \int_0^1 \frac{F_t}{F_x} dF dt = 0 \tag{5.9}$$

since F is $1-$periodic in t. Subtracting (5.9) from (5.8), using (5.6) and the $1-$ periodicity of X w.r. to F we obtain:

$$r = \int_0^1 \int_0^1 S_{-\Lambda}\overline{X}_t(F,t) dF dt = \int_0^1 \int_0^1 \overline{X}_t(F,t) dF d \ . \tag{5.10}$$

Now, let $\mu^{(i)} \in \mathbf{H}_C^\infty$, $i = 1, 2$, the corresponding shifted inverse of the cumulation functions $\overline{X}^{(i)}(F,t)$ and shifts Λ_i. Let the inner product

$$\langle \mu_1, \mu_2 \rangle = \int_0^1 \int_0^1 S_{-\Lambda_1}\left(\overline{X}_t^{(1)}\right) \cdot S_{-\Lambda_2}\left(\overline{X}_t^{(2)}\right) dF dt \tag{5.11}$$

and the metric

$$D(\mu^{(1)}, \mu^{(2)}) = \sqrt{\int_0^1 \int_0^1 \left|S_{-\Lambda_1}\left(\overline{X}_t^{(1)}\right) - S_{-\Lambda_2}\left(\overline{X}_t^{(2)}\right)\right|^2 dF dt} + \sup_{t \in [0,1]} W_2\left(\mu_t^{(1)}, \mu_t^{(2)}\right) \ . \tag{5.12}$$

It follows from Lemma 5.1 that (5.11) and (5.12) are consistent with the inner product and metric defined in Definition 5 where γ is replaced by the Lebesgue measure, provided $X^{(i)}$ are chosen in the standard gauge.

On the other hand, if we apply (5.10) to $X^{(i)}$ in the standard gauge we obtain

$$|r_1 - r_2| = \left| \int_0^1 \int_0^1 \left(\overline{X}_t^{(1)} - \overline{X}_t^{(2)} \right) dF dt \right| \leq D\left(\mu^{(1)}, \mu^{(2)}\right) ,$$

hence

Lemma 5.3. *The rotation number defined on $\mathbf{H}_C^\infty$ is $1-$Lipschitz with respect to the D topology.*

We now use Theorem 2.2 to close $\mathbf{H}_C^\infty$ in the *strong* topology:

Theorem 5.1. *The rotation number r is defined, continuous and $1-$Lipschitz on the whole of $\mathbf{H}_{C,2}$ with respect to the strong topology induced by D.*

Let us consider again the cumulation inverse X corresponding to $\mu \in \mathbf{H}_C^\infty$. Since $\Xi(F,t) := X(F,t) - F$ is a periodic function of F, we can expand it into a Fourier series:

$$\Xi(F,t) = \sum_{j \in \mathbb{Z}} a_j(t) e^{2\pi i j F} .$$

It follows that, for X in the standard gauge

$$S_A X(F,t) = F + A(t) + \sum_{j\in\mathbb{Z}} a_j(t) e^{2\pi i j (F + A(t))} \tag{5.13}$$

also

$$\overline{X}_t(F,t) := \left(S_\Lambda X\right)_t(F,t) = \dot{\Lambda}(t) + \sum_{k\in\mathbb{Z}} \left[\dot{a}_k + 2\pi i k \dot{\Lambda} a_k\right] e^{2\pi i k (F + \Lambda(t))} \tag{5.14}$$

and

$$S_{-\Lambda}\overline{X}_t(F,t) = \dot{\Lambda}(t) + \sum_{k\in\mathbb{Z}} \left[\dot{a}_k + 2\pi i k \dot{\Lambda} a_k\right] e^{2\pi i k F} \tag{5.15}$$

We obtain the following:

Proposition 5.1. *Let $\mu^{(j)} \in \mathbf{H}_C^\infty$ converges weakly to $\mu \in \mathbf{H}_{2,C}$. Then the corresponding inverse cumulations X_j in the standard gauge converges strongly in $\mathbb{L}^2([0,1]^2)$ to the inverse cumulation X of μ. If, in addition, the convergence on $\mu^{(j)}$ is strong, then*

- *i) If $\Lambda_j(t)$ is the corresponding shifts, then there exists $\Lambda = \Lambda(t)$ such that $\lim_{j\to\infty} \int_0^1 |\dot{\Lambda}_j(t) - \dot{\Lambda}(t)|^2 dt = 0$.*
- *ii) The sequence $\partial_t \overline{X}_j$ converges in strong $\mathbb{L}^2([0,1]^2)$ to $Z := \partial_t S_\Lambda X$. In particular, $Z \in \mathbb{L}^2([0,1]^2)$.*

Remark: Note that Proposition 5.1 does not guarantee, in general, that Λ is the optimal shift corresponding to the limit X in the sense of Lemma 5.2, nor that such an optimal shift exists at all.

Proof. From Definition 2 and (5.3) we obtain the $\{\mu^{(j)}\}$ is a Cauchy sequence in $C([0,1],C^*)$ where C^* is equipped with the W_2 metric. In particular there exists a subsequence for which

$$\sum_{j=1}^{\infty} W_2\left(\mu^{(j+1)},\mu^{(j)}\right) < \infty \ .$$

This implies that there exists a sequence $\alpha_j = \alpha_j(t)$ on $[0,1]$ such that

$$\sum_{j=1}^{\infty} \int |X_{j+1}(F+\alpha_{j+1}(t),t) - X_j(F,t)|^2 \, dFdt < \infty$$

where X_j are the inverse cumulations of $\mu^{(j)}$. Let $A_j(t) = \sum^j \alpha_i(t)$. Then

$$\sum_{j=1}^{\infty} \int \left|S_{A_{j+1}}X_{j+1} - S_{A_j}X_j\right|^2 dFdt < \infty$$

so the subsequence $S_{A_j}X_j$ converges $\mathbb{L}^2$ strongly to some $Y \in \mathbb{L}^2([0,1]^2)$.

From (5.13) it follows that

$$\left\|S_{A_{j+1}}X_{j+1} - S_{A_j}X_j\right\|_2^2 =$$
$$\int_0^1 |A_{j+1}(t) - A_j(t)|^2 \, dt + \sum_{k\in\mathbb{Z}} \int_0^1 \left|a_k^{(j+1)}(t)e^{2\pi i k A_{j+1}(t)} - a_k^j(t)e^{2\pi i k A_j(t)}\right|^2 dt$$

where $a_k^{(j)}$ is a Fourier coefficient of $X_j - F$. In particular, A_j is also a Cauchy sequence in $\mathbb{L}^2$ and converges strongly to some $A \in \mathbb{L}^2([0,1])$.

It implies, in particular, that for any $\phi \in C^1([0,1]^2)$, the sequence $\phi_j := S_{A_j}\phi$ converges $\mathbb{L}^2$ strongly to $S_A\phi$.

Now, we can further restrict the subsequence so that X_j converges in distribution to some $X \in \mathbb{L}^2([0,1]^2)$. It follows that

$$\int_0^1 \int_0^1 X_j \phi dFdt \to \int_0^1 \int_0^1 X\phi dFdt \ . \tag{5.16}$$

On the other hand

$$\int_0^1 \int_0^1 X_j\phi dFdt = \int_0^1 \int_0^1 S_{A_j}(X_j)\phi_j dFdt \to \int_0^1 \int_0^1 Y S_A(\phi) dFdt = \int_0^1 \int_0^1 S_{-A}(Y)\phi dFdt \tag{5.17}$$

since both subsequences $S_{A_j}X_j$ and ϕ_j converge strongly to Y and $S_A\phi$, respectively.

From (5.16) and (5.17) we obtain that $X = S_{-A}Y$. In particular:

$$\|X\|_2 = \|Y\|_2 = \lim_{n\to\infty} \|S_{A_n}X_n\|_2 = \lim_{n\to\infty} \|X_n\|_2$$

where the second equality follows from the strong convergence of $S_{A_j}X_j$ to Y. This, together with the assumed weak convergence of X_j to X, implies the strong convergence of this sequence.

Now, if another subsequence admits another limit $\hat{X}$, then, by the same argument as before, $\hat{X}$ is a shift of X, namely $\exists C = C(t)$ such that $\hat{X} = S_C X$. But since both X and $\hat{X}$ admit the standard gauge (which is preserved in the weak limit), it follows that $C \equiv 0$.

i) From (5.14) and (5.12) it follows that the sequence $S_{-\Lambda_j}\partial_t\overline{X}_j$ is Cauchy in the strong $\mathbb{L}^2$ sense, and that

$$\left\|S_{-\Lambda_j}\partial_t\overline{X}_j - S_{-\Lambda_m}\partial_t\overline{X}_m\right\|_2^2 = \int_0^1 \left|\dot{\Lambda}_j(t) - \dot{\Lambda}_m(t)\right|^2 dt + \sum_{k\in\mathbb{Z}}\int_0^1 \left|\beta_k^{(j)} - \beta_k^{(m)}\right|^2 dt$$

where $\beta_k^{(j)} := \dot{a}_k^{(j)} - 2\pi ik\dot{\Lambda}_j a_k^{(j)} e^{-2\pi ik\Lambda_j(t)}$. In particular, $\dot{\Lambda}_j$ is a Cauchy sequence in $\mathbb{L}^2([0,1])$ as well.

ii) The proof of (ii) is the same as the proof of the first part of the Proposition, using the strong $\mathbb{L}^2$ convergence of Λ_j, implied by (i).

$\square$

5.2 Weak continuity

In section 6 we demonstrate the existence of a weakly converging sequence $\mu^{(j)} \in \mathbf{H}_C^\infty$ such that the rotation number of its limit does not coincide with the limit of the rotation numbers. In the case under consideration, the densities ρ_j associated with $\mu^{(j)}$ satisfy $\int_0^1\int_0^1 \rho_j^{-1}dxdt = \infty$ while the density of the weak limit satisfies $\int_0^1\int_0^1 \rho^{-1} < \infty$.

Note that, in terms of the inverse cumulation X:

$$||\rho^{-1}||_1 = \int_0^1\int_0^1 \rho^{-1}dxdt = \int_0^1\int_0^1 F_x^{-1}dxdt = \int_0^1\int_0^1 |X_F|^2 dFdt \ . \qquad (5.18)$$

Lemma 5.4. *Let $\mu \in \mathbf{H}_{C,2}$ such that the corresponding density ρ satisfies $\rho^{-1} \in \mathbb{L}^1$. Then the following are equivalent*

i) $\quad \dot{\Lambda}(t) = -\frac{\int_0^1 X_F X_t dF}{\int_0^1 |X_F|^2 dF}$.

ii) $A = \Lambda$ *is the minimizer of* $\int_0^1\int_0^1 \left|\partial_t\left(S^A X\right)\right|^2 dFdt$.

iii) For any continuous, periodic function $b = b(t)$,
$\int_0^1\int_0^1 b(t)\partial_t\left(S^\Lambda X\right)\partial_F\left(S^\Lambda X\right)dFdt = 0$.

The rotation number $r = r(\mu)$ is determined by $r = \Lambda(1) - \Lambda(0)$.

Proof. First note that, by (5.18), (5.7) and the Cauchy-Schwartz inequality that $\dot{\Lambda} \in \mathbb{L}^1([0,1])$. Let B be any differentiable functions whose derivatives $b = B'$ is continuous and periodic. The equivalence of (i,ii,iii) follows immediately from

$$\int_0^1\int_0^1 \left|\partial_t\left(S^{\Lambda+\varepsilon B}X\right)\right|^2 dFdt = \int_0^1\int_0^1 \left|\partial_t\left(S^\Lambda X(F+\varepsilon B(t),t)\right)\right|^2 dFdt$$

$$= \int_0^1\int_0^1 \left|\varepsilon\partial_F\left(S^\Lambda X\right)(F+\varepsilon B(t),t)b(t) + \partial_t\left(S^\Lambda X(F+\varepsilon B,t)\right)\big|_{B=B(t)}\right|^2 dFdt$$

$$= \int_0^1 \int_0^1 \left|\partial_t \left(S^\Lambda X\right)(F,t)\right|^2 dFdt + 2\varepsilon \int_0^1 \int_0^1 \partial_t \left(S^\Lambda X\right) \partial_F \left(S^\Lambda X\right) bdFdt$$

$$+\varepsilon^2 \int_0^1 \int_0^1 \left|\partial_F \left(S^\Lambda X\right)\right|^2 b^2(t) dFdt \; . \tag{5.19}$$

Finally, let $\mu_n \in \mathbf{H}_C^\infty$ converge strongly to μ. Let Λ_n be the shift associated with μ_n. Then Proposition 5.1 guarantees that the strong $\mathbb{L}^2$ convergence of $\dot{\Lambda}_n$ to $\dot{\Lambda}$. The last statement follows from (5.8) which implies $r(\mu_n) = \Lambda_n(1) - \Lambda_n(0)$, from $\Lambda_n(1) - \Lambda_n(0) \to \Lambda(1) - \Lambda(0)$ and from the continuity of r in the strong topology. □

The main result of this section is

Theorem 5.2. *Let $K > 0$ and $\mathbf{K} \subset \mathbf{H}_C^\infty$ such that $\partial_x \rho^{-1} \in \mathbb{L}^1(\mathbb{S}^1 \times \mathbb{S}^1)$ and*

$$\left\| \frac{\partial \rho^{-1}}{\partial x} \right\|_1 < K \quad , \tag{5.20}$$

for any density of $\mu \in \mathbf{K}$. Let $\overline{\mathbf{K}}$ the weak closure of $\mathbf{K}$. Then, the rotation number is continuous with respect to the weak topology on $\overline{\mathbf{K}}$.

Before proceeding, we interpret the condition (5.20) in terms of the inverse cumulation X as follows:

$$\left\| \frac{\partial \rho^{-1}}{\partial x} \right\|_1 = \int_0^1 \int_0^1 \frac{|\rho_x|}{\rho^2} dxdt = \int_0^1 \int_0^1 |X_{FF}| \, dFdt \; . \tag{5.21}$$

Indeed, if we differentiate the identity $F_x X_F = 1$ by the variable F we obtain

$$F_{xx}|X_F|^2 + F_x X_{FF} = 0 \;\to F_{xx} = -\frac{F_x X_{FF}}{|X_F|^2} = -\frac{X_{FF}}{|X_F|^3}$$

and (5.21) follows from $|\rho_x|\rho^{-2}dx = |F_{xx}|F_x^{-2}dx = |F_{xx}|X_F^3 dF$.

From (5.21) and (5.2) we obtain

Lemma 5.5. *Assume $\mu \in \mathbf{H}_C^\infty$ with the associated density ρ satisfying (5.20). Let X be the associated inverse cumulation in standard gauge. Then the Fourier coefficients a_j of $\Xi = X - F$ satisfy*

$$\int_0^1 |a_k(t)| dt \le \frac{K}{(2\pi)^2 |k|^2} \;\; \forall k \neq 0 \quad ; \quad a_0(t) = \frac{1}{2} \quad \forall t \; .$$

Next, from

$$\overline{X}(F,t) = \Xi(F + \Lambda(t), t) + F + \Lambda(t)$$

we obtain the Fourier expansion of $\overline{X}_t$ and $\overline{X}_F$ (both periodic in F) via

$$\overline{X}_t = \dot{\Lambda} + \sum_{0 \neq k \in \mathbb{Z}} \left[\dot{a}_k + 2\pi i k a_k \dot{\Lambda}\right] e^{2\pi i k(F+\Lambda(t))} \; , \; \overline{X}_F = 1 + 2\pi i \sum_{k \in \mathbb{Z}} k a_k e^{2\pi i k(F+\Lambda(t))} \tag{5.22}$$

where we used $a_0 = 1/2$, hence $\dot{a}_0 = 0$ in the standard gauge. It follows from Lemma 5.4-(iii) that

Lemma 5.6. *Given $\mu^{(n)} \in \mathbf{H}_C^\infty$, X_n the corresponding inverse cumulation in standard gauge and Λ_n the corresponding shift, then*

$$\int_0^1 |\dot{\Lambda}_n|^2 dt + \sum_{0\neq k\in\mathbb{Z}} \int_0^1 |\dot{a}_k^{(n)} + 2\pi ik\dot{\Lambda}_n a_k^{(n)}|^2 dt = ||\mu||_2^2 \tag{5.23}$$

as well as

$$\int_0^1 b(t) \left[\dot{\Lambda}_n + \sum_{k\in\mathbb{Z}} k\overline{a}_k^{(n)} \dot{a}_k^{(n)} + 2\pi i \sum_{k\in\mathbb{Z}} k|a_k^{(n)}|^2 \dot{\Lambda}_n \right] dt = 0 \tag{5.24}$$

for any continuous, periodic $b = b(t)$.

Proof. of Theorem 5.2:
We know by Proposition 5.1 the strong $\mathbb{L}^2$ convergence of $X_n \to X$. In addition, $\partial_F X_n$ are bounded uniformly in $\mathbb{L}^2([0,1]^2)$ since ρ_n^{-1} are bounded uniformly in $\mathbb{L}^1([0,1]^2)$, so $X_F \in \mathbb{L}^2([0,1]^2)$ as well. Since $\|\mu^{(n)}\|_2$ are uniformly bounded, it follows from (5.23) that $\dot{\Lambda}_n$ are uniformly bounded in $\mathbb{L}^2([0,1])$, so there exists a subsequence along which $\dot{\Lambda}_n \to \dot{\Lambda}$ in the weak $\mathbb{L}^2$ sense, and $\Lambda_n \to \Lambda$ uniformly. Since $r(\mu^{(n)}) = \Lambda_n(1) - \Lambda_n(0)$, it is enough to show that Λ is the optimal shift associated with the limit X, hence $r = \Lambda(1) - \Lambda(0)$. By Lemma 5.4-(iii) it is enough to show that $\dot{\Lambda} + \Psi \equiv 0$ as an $\mathbb{L}^2$ function, where

$$\Psi := \sum_{k\in\mathbb{Z}} k\overline{a}_k \dot{a}_k + 2\pi i \sum_{k\in\mathbb{Z}} k|a_k|^2 \dot{\Lambda} \ ,$$

a_k are the Fourier coefficients of X.

For any $k \in \mathbb{Z}$, the strong $\mathbb{L}^2([0,1]^2)$ convergence of X_n to X implies the strong $\mathbb{L}^2([0,1])$ convergence of the Fourier coefficient $a_k^{(n)}$ of X_n to a_k. Since $\dot{\Lambda}_n$ converges weakly to $\dot{\Lambda}$ as well, then $2\pi ik\dot{\Lambda}_n a_k^{(n)}$ converges $\mathbb{L}^2$ weakly to $2\pi ik\dot{\Lambda} a_k$, and $2\pi ik\dot{\Lambda}_n \left| a_k^{(n)} \right|^2$ converges $\mathbb{L}^2$ weakly to $2\pi ik\dot{\Lambda} \left| a_k \right|^2$. In addition, we have by (5.23) the weak $\mathbb{L}^2([0,1])$ convergence (along a subsequence) of $\dot{a}_k^{(n)} + 2\pi ik\dot{\Lambda}_n a_k^{(n)}$. This implies that $\dot{a}_k^{(n)}$ converges $\mathbb{L}^2$ weakly as well, and its limit must be $\dot{a}_k$. In particular, $\overline{a}_k^{(n)} \dot{a}_k^{(n)}$ converges $\mathbb{L}^2$ weakly to $\overline{a}_k \dot{a}_k$ for any k.

We obtain from the above that, for any $N \in \mathbb{N}$, the weak $\mathbb{L}^2$ convergence, as $n \to \infty$, of

$$\Psi_N^{(n)} := \sum_{|k|\leq N} k\overline{a}_k^{(n)} \dot{a}_k^{(n)} + 2\pi i \sum_{|k|\leq N} k|a_k^{(n)}|^2 \dot{\Lambda}_n$$

to

$$\Psi_N := \sum_{|k|\leq N} k\overline{a}_k \dot{a}_k + 2\pi i \sum_{|k|\leq N} k|a_k|^2 \dot{\Lambda} \ .$$

Let

$$\Psi^{(n)} := \sum_{k\in\mathbb{Z}} k\overline{a}_k^{(n)} \dot{a}_k^{(n)} + 2\pi i \sum_{k\in\mathbb{Z}} k|a_k^{(n)}|^2 \dot{\Lambda}_n \ .$$

From Lemma 5.5 and (5.23) we obtain

$$\left\| \Psi^{(n)} - \Psi_N^{(n)} \right\|_2 \leq O(N^{-1/2})$$

so $\Psi^{(n)} \to \Psi$ in the weak $\mathbb{L}^2$ norm. Then, (5.24) implies that $\dot{\Lambda} + \Psi \equiv 0$ (as an $\mathbb{L}^2([0,1])$ function), and the weak limit Λ is an optimal shift.

Finally, the uniqueness of the optimal shift function Λ, implied by Lemma 5.4, guarantees that the limit Λ is independent of the subsequence, and the weak convergence of Λ_n follows. $\square$

6 Examples revisited

We shall first consider Example 4 of the introduction:

Rigid rotation μ: A rigid rotation is characterized by

$$\mu_{t+\sigma} = \left[R_{X(t+\sigma)-X(t)}\right]_{\#} \mu_t \quad \forall t, \sigma \in \mathbb{R} \tag{6.1}$$

where $R_\sigma(x) =: x + \sigma \mod 1$ and $X(t)$ is a deterministic, continuous orbit with a prescribed topological degree d. If, in addition, $\mu \in \mathbf{H}_C^\infty$ then (6.1) corresponds to a density of the form $\rho(x,t) = g(x - X(t))$ with g a positive, smooth probability density on $\mathbb{R}^1 \mod \mathbb{Z}$ and X a smooth orbit. Let us compute the rotation number associated with such μ. In the special gauge $F(0,t) = 0$ the cumulation function F is given by

$$F = G(x - X(t)) - G(-X(t))$$

where G is the primitive of g. Thus

$$\langle F_t(\cdot,t)\rangle_{\rho^{-1}(\cdot,t)} = \dot{X}(t)\frac{\int_0^1 \frac{g(-X(t)) - g(x-X(t))}{g(x-X(t))}dx}{\int_0^1 g^{-1}(x - X(t))dx} = \dot{X}(t)\left(g(-X(t)) - \frac{1}{\int_0^1 g^{-1}(x)dx}\right) . \tag{6.2}$$

Recall

$$d = \int_0^1 \dot{X}(t)dt .$$

Since $G(\cdot)$ is also a cumulation function, then:

$$d = \int_0^1 \frac{dG}{dt}(X(t))dt = -\int_0^1 \frac{dG}{dt}(-X(t))dt = \int_0^1 \dot{X}(t)g(-X(t))dt$$

hence

$$r = \int_0^1 \langle F_t(\cdot,t)\rangle_{\rho^{-1}(\cdot,t)}dt = d\left(1 - \frac{1}{\int_0^1 g^{-1}}\right) . \tag{6.3}$$

Since g is a normalized density, $\int_0^1 g^{-1} \geq 1$ always, where equality holds only if $g \equiv 1$ (the uniform Lebesgue measure). We thus proved

Proposition 6.1. *For any rigid rotation $\mu_t \in \mathbf{H}_C^\infty$, the rotation number r is in the interval $[0,d)$ if $d > 0$, $(d,0]$ if $d < 0$, where d is the degree of the deterministic orbit $t \to X(t)$. In addition, $r = 0$ iff either $d = 0$ or $g \equiv 1$.*

We shall now prove that the rotation number is *not* continuous with respect to weak convergence. Let g be a strictly positive and smooth density on $\mathbb{S}^1$, identified with a 1-periodic density function on $\mathbb{R}$. Let $\{g_j\}$ a sequence of smooth and positive densities such that

$$g_j(x) = g(x) \text{ for } \ x \in \mathbb{R}/(0, 1/j) \mod \mathbb{Z}$$

and

$$\lim_{j\to\infty} \int_0^1 \frac{1}{g_j}(x)dx = \infty \ .$$

Let $\rho_j(x,t) = g_j(x - X(t))$, $\rho(x,t) = g(x - X(t))$ and μ_j, μ the corresponding orbits, where X is a smooth, deterministic circle orbit with a topological degree $d \in \mathbb{Z}$. A direct computation, similar to (6.2), yields

$$\left\langle F_t^2 \right\rangle_{\rho^{-1}} = |\dot{X}|^2 \left[g^2(-X(t)) - 2\frac{g(-X(t))}{\int_0^1 g^{-1}(x)dx} + \frac{1}{\int_0^1 g^{-1}(x)dx} \right] \tag{6.4}$$

and correspondingly for μ_j. Using (6.2), (6.4) and (4.5) we obtain

$$||\mu||_2 = \sqrt{1 - \frac{1}{\int_0^1 g^{-1}(x)dx}} \sqrt{\int_0^1 |\dot{X}(t)|^2 dt} \ \ ; \ \ ||\mu_j||_2 = \sqrt{1 - \frac{1}{\int_0^1 g_j^{-1}(x)dx}} \sqrt{\int_0^1 |\dot{X}(t)|^2 dt} \ . \tag{6.5}$$

It follows that

$$\lim_{j\to\infty} ||\mu_j||_2 = \sqrt{\int_0^1 |\dot{X}(t)|^2 dt} > ||\mu||_2.$$

In particular $\{\mu_j\}$ are uniformly bounded in $\mathbf{H}_{C,2}$. The convergence of μ_j to μ in $C\left([0,1], C^*(\mathbb{S}^1)\right)$ is evident, hence the weak convergence of μ_j to μ is established via Definition 2.

On the other hand, by (6.3), $\lim_{j\to\infty} r(\mu_j) = d$, while $r(\mu) \neq d$. Since d can be either positive or negative integer it follows that the rotation number is neither upper nor lower semi-continuous with respect to weak convergence.

We shall now revisit Example 1 of the introduction, and ask whether the rotation number so defined contains the topological degree of a deterministic, continuous circle orbit $X : \mathbb{S}^1 \to \mathbb{S}^1$? It turns out that it is, indeed, the case, provided $\dot{X} \in \mathbb{L}^2(\mathbb{S}^1)$.

Proposition 6.2. *Assume $\int_0^1 |\dot{X}(t)|^2 dt < \infty$. Let $\mu_t = \delta_{x-X(t)}$. Then $\mu = \mu_t dt \in \mathbf{H}_{C,2}$ and the rotation number of μ is an integer, coinciding with the topological degree of the continuous orbit X.*

Proof. Let g_j be a sequence of smooth, strictly positive 1-periodic densities on $\mathbb{R}$ such that

$$\lim_{j\to\infty} g_j = \sum_{i=-\infty}^{\infty} \delta_{x-i} \tag{6.6}$$

as distributions. Let X_j be a sequence of smooth orbits which converges strongly to X, namely

$$\lim_{j\to\infty} \int_0^1 |\dot{X}_j(t) - \dot{X}(t)|^2 dt = 0 \ .$$

Let G_j the cumulation functions corresponding to g_j, and $G_j(x - X_j(t))$ the cumulation functions of the sequence of rigid rotations. Evidently, $\mu_j =: g_j(x - X_j(t))dxdt \in \mathbf{H}_C^\infty$ converges to $\mu = \delta_{X(t)-x}dt$ uniformly in $C\left([0,1]; C^*(\mathbb{S}^1)\right)$. Since $\lim_{j\to\infty} \int_0^1 g_j^{-1}(x)dx = \infty$ we obtain from (6.5) that $\lim_{j\to\infty} ||\mu_j||_2^2 = \int_0^1 |\dot{X}|^2 dt$. In particular, $||\mu_j||_2$ are uniformly bounded and $\mu_j \to \mu$ in the weak topology.

Moreover from (2.1) and (2.2) we obtain that $||\mu||_2^2 = \int_0^1 |\dot{X}|^2 dt$. Lemma 2.5 implies that $\mu_j \to \mu$ in the *strong* topology as well. By Theorem 5.1 we obtain the convergence of $r(\mu_j)$ to $r(\mu)$. Since μ_j are rigid rotations and $\int_0^1 g_j^{-1}(x)dx = \infty$ we obtain, as in the proof of Proposition 6.1, that $r(\mu_j)$ converges to the degree of X. $\square$

We now proceed to a special case of Example 3:

Proposition 6.3. *Let*

$$\mu_t = \sum_i^N \beta_i \delta_{(X_i(t)-x)}$$

where X_i are C^1 orbits of degree $d_i \in \mathbb{Z}$ and $\sum_1^N \beta_i = 1$, $\beta_i > 0$. Then $\mu \in \mathbf{H}_{C,2}$ and $r(\mu) = \sum \beta_i d_i$.

Proof. As in Proposition 6.2 we can show that $\mu \in \mathbf{H}_{C,2}$ and

$$||\mu||_2^2 = \sum_1^N \beta_i \int_0^1 \left|\dot{X}_i\right|^2 dt \quad . \tag{6.7}$$

Let $G_{j,x} = g_j$ be as in the proof of Proposition 6.2, so, in the special gauge $F(0,t) = 0$,

$$F_j(x,t) = \sum_{i=1}^N \beta_i \left[G_j(x - X_i(t)) - G_j(-X_i(t))\right]$$

be a sequence of cumulation functions and

$$\rho_j(x,t) = \sum_{i=1}^N \beta_i g_j(x - X_i(t))$$

the corresponding densities. Proceeding similarly to (6.2) we obtain

$$\begin{aligned}
\langle F_{j,t}(\cdot,t)\rangle_{\rho_j^{-1}(\cdot,t)} &= \sum_i \beta_i \dot{X}_i(t) \frac{\int_0^1 \frac{g_j(-X_i(t)) - g_j(x - X_i(t))}{\rho_j(x,t)} dx}{\int_0^1 \rho_j^{-1}(x,t)dx} \\
&= \sum_i \beta_i \dot{X}_i(t) \left[g_j(-X_i(t)) - \frac{\int_0^1 \frac{g_j(x-X_i(t))}{\rho_j(x,t)} dx}{\int_0^1 \rho_j^{-1}(x,t)dx} \right] \quad .
\end{aligned} \tag{6.8}$$

Next we note that $\int_0^1 g_j(x - X_i(t))/\rho_j(x,t)dx$ is uniformly bounded in t and $\int_0^1 \rho_j^{-1}(x,t)dx \to \infty$ uniformly as $j \to \infty$. Then, as $j \to \infty$:

$$r(\mu_j) = \int_0^1 \langle F_{j,t}\rangle \, dt = \sum_i \beta_i \int_0^1 \dot{X}_i(t) g_j(-X_i(t))dt + o(1) = \sum_i \beta_i d_i + o(1) \tag{6.9}$$

as in (6.3). Next, we prove that μ_j converges in the strong D topology. For this we estimate

$$\langle F_{j,t}^2\rangle = \sum_{i,k}\beta_i\beta_k\dot{X}_i(t)\dot{X}_k(t)\left\{g_j(-X_i(t))g_j(X_k(t)) + \frac{1}{\int_0^1\rho_j^{-1}(x,t)dx}\right.$$

$$\left.\left[\int_0^1 \frac{g_j(x-X_i(t))g_j(x-X_k(t))}{\rho_j(x,t)}dx - g_j(-X_i)\int_0^1\frac{g_j(x-X_k(t))}{\rho_j(x,t)}dx\right]\right\}$$

hence, by (4.5) and (6.8)

$$||\mu||_2^2 = \int_0^1\sum_{i,k}\beta_i\beta_k\dot{X}_i(t)\dot{X}_k(t)\left\{\int_0^1\frac{g_j(x-X_i(t))g_j(x-X_k(t))}{\rho_j(x,t)}dx\right.$$

$$\left.-\left(\int_0^1\rho_j^{-1}(x,t)dx\right)^{-1}\int_0^1\frac{g_j(x-X_i(t))}{\rho_j(x,t)}dx\int_0^1\frac{g_j(x-X_k(t))}{\rho_j(x,t)}dx\right\}dt\ . \tag{6.10}$$

Now, the estimates we got, preceding (6.9), on $\int_0^1 g_j(x-X_i(t))/\rho_j(x,t)dx$ and $\int_0^1\rho_j^{-1}(x,t)dx$ kill the second term in (6.10) in the limit $j\to\infty$. As for the first term, we observe that $\rho_j^{-1}(x,t)g_j(x-X_i(t))$ is, on the one hand, uniformly bounded with respect to j and t as a function of x and for some $\delta_j>\varepsilon_j$, both converging to 0:

$$\lim_{j\to\infty}\sup_{|x-X_i(t)|\le\varepsilon_j}\left|\rho_j^{-1}(x,t)g_j(x-X_i(t))-\beta_i^{-1}\right| = 0$$

and

$$\lim_{j\to\infty}\sup_{|x-X_i(t)|\ge\delta_j}\rho_j^{-1}(x,t)g_j(x-X_i(t)) = 0$$

uniformly in t. From these and (6.6) we obtain

$$\int_0^1\int_0^1\frac{g_j(x-X_i(t))g_j(x-X_k(t))}{\rho_j(x,t)}dxdt = \beta_i^{-1}\delta_{i,k}\ ,$$

so (6.10) yields

$$\lim_{j\to\infty}||\mu_j||_2^2 = \sum_i\beta_i\int_0^1\left|\dot{X}_i(t)\right|^2dt\ .$$

From (6.7) and Lemma 2.5 it follows that $\mu_j\to\mu$ in the *strong* D topology. Hence, Theorem 5.1 implies that $r(\mu_j)\to r(\mu)$. Finally, (6.9) completes the proof. □

7 Conclusions and open problems

We attempted a generalization of topological degree for (probabilistic) measure valued circle mappings μ. It was done by restriction to those mappings for which we could associate some weak notion of flow. Our treatment, however, is not complete because the associated velocity field is assumed to be an $\mathbb{L}^2$ function with respect to the measure μ. Since there are continuous mappings $X : \mathbb{S}^1 \to \mathbb{S}^1$ for which $\int_{\mathbb{S}^1} |\dot{X}|^2 dt$ is not defined, we cannot extend our definitions to measures like $\delta_{x-X(t)} dt$ of the above form. The first question is:

Q₁ *Can the rotation number be extended to include all orbits $\mu \in C\left(\mathbb{S}^1; C^*(\mathbb{S}^1)\right)$? Can it be extended further beyond this class to include, say, the measure valued VMO (see, e.g., [BN])?*

The second question is related to the actual existence of a flow for the space $\mathbf{H}_{C,2}$. We know that a velocity field $\boldsymbol{v}$ exists in $\mathbb{L}^2[\mu]$ by definition. In which sense the flow $\dot{x} = \boldsymbol{v}(x,t)$ can be defined, such that $\lim_{t\to\infty} t^{-1}x(t)$ is consistent with the rotation number? Can it always be defined as a deterministic flow?

To demonstrate this point, let us reconsider Example (3) of the introduction. Assume that all intersections are binary, that is, if t^* is the intersection time of orbits i and j, i.e. $X_i(t^*) = X_j(t^*)$, then $X_k(t^*) \neq X_i(t^*)$ for any $k \neq i, j$. We may associate a stochastic flow $Z(t)$ on the support of this measure as follows:
Let $Z(t_0) = X_i(t_0)$ at t_0 which is not an intersection time of the orbit X_i with any other orbit, and X_j is the *first* orbit intersection X_i after t_0. Let $t^* > t_0$ be the time of this intersection. Then $Z(t) = X_i(t)$ for $t_0 \leq t \leq t^*$ while $Z(t) = X_j(t)$ for a right neighborhood of t^* with probability $p = \min\{1, \beta_j/\beta_i\}$, $Z(t) = X_i(t)$ otherwise.

With the above definition we obtain $Z(t)$ as a parameterized family of random variables. The question we address is:

Q₂ *Is $\lim_{t\to\infty} t^{-1}Z(t) = \sum \beta_j d_j$ with probability 1?*

Notice that, in the special case $\beta_i = 1/N$ for all $i = 1, 2, \ldots n$, the above process is a deterministic one. It can easily be proved by elementary arguments that, in this case, the answer to **Q₂** is positive.

The next question concerns generalizations of the results of this paper to higher dimension. Here we may think about several possible directions.
One possibility is to consider orbits of periodic measures supported on the $n-$ torus $\mathbb{T}^n$. Hence $\mu = \mu_t dt$ with $\mu_0 = \mu_1$ and, for each t, μ_t is a probability measure on $\mathbb{T}^n$. Note that for a deterministic orbit $X : \mathbb{S}^1 \to \mathbb{T}^n$, a degree $\boldsymbol{d} \in \mathbb{Z}^n$ is defined.

Q₃ *Is it possible to extend this definition to a rotation vector $\mathbf{r} \in \mathbb{R}^n$ for $\mu \in \mathbf{H}_{C,2}(\mathbb{T}^n)$?*

Finally, we mention again the conjecture suggested at the introduction:

Q₄ *Can the condition of Theorem 5.2 be relaxed to $\|\rho^{-1}\|_1 < K$?*

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