Connecting the Deep Quench Obstacle Problem with Surface Diffusion via their Steady States

In memory of Maria Conceicao Carvalho and her passion for life and science.

Eric A. Carlen, Amy Novick-Cohen, and Lydia Peres Hari

Abstract In modeling phase transitions, it is useful to be able to connect diffuse interface descriptions of the dynamics with corresponding limiting sharp interface motions. In the case of the deep quench obstacle problem (DQOP) and surface diffusion (SD), while a formal connection was demonstrated many years ago, rigorous proof of the connection has yet to be established. In the present note, we show how information regarding the steady states for both these motions can provide insight into the dynamic connection, and we outline tools that should enable further progress. For simplicity, we take both motions to be defined on a planar disk.

Key words: Deep quench obstacle problem, surface diffusion, higher order degenerate parabolic equations, geometric motions, limiting motions.

1 Introduction

Many two component mixtures exist stably at one temperature, but become unstable at lower temperatures; the subsequent instability typically initiates phase separation, leading to the appearance of spatial regions characterized by two different compositional phases. During the early stages of phase separation, the distinction between the two phases is not sharp and a diffuse description is appropriate. As phase separation progresses, the phases become more distinct, and a sharp interface description is appropriate. Accordingly, in physically realistic models, it should be possible to pass

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from one description to the other. Unfortunately, often there is a gap between what is physically reasonable and what is possible to justify with mathematical rigor. For example, though the diffuse interface Cahn-Hilliard model with a logarithmic potential and a degenerate mobility has been shown using formal asymptotics [6] to yield the sharp interface surface diffusion model, the connection has yet to have been made rigorous. In the present note, we focus on the zero temperature limit of the Cahn-Hilliard model with a logarithmic potential and degenerate mobility, namely on the deep quench obstacle problem with degenerate mobility, which we shall subsequently refer to here simply as the deep quench obstacle problem (DQOP), and its connection with motion by (isotropic) surface diffusion (SD).

After recalling below some relevant background with regard to both models, in Section 2 we discuss the steady states in some detail for both models, focusing in particular on the minimum energy steady states, and then in Section 3 we outline certain tools that we are using to bridge the two evolutions.

Let us consider both motions, (DQOP) and (SD), to be defined in $\Omega \subset R^2$, where Ω is a disk centered at the origin whose radius, R_0, is $O(1)$. In considering time evolution, we set $\Omega_T := \Omega \times (0,T)$ and $\partial\Omega_T := \partial\Omega \times (0,T)$ with $0 < T \leq \infty$. The deep quench obstacle problem [19] with the degenerate mobility $M(u) = 1 - u^2$, can be expressed as

$$\text{(DQOP)} \quad \begin{cases} u_t = \nabla \cdot M(u)\nabla w, \quad w + u + \varepsilon^2 \triangle u \in \partial I_{[-1,1]}(u), \quad (x,t) \in \Omega_T, \\ n \cdot \nabla u = n \cdot M(u)\nabla w = 0, \quad (x,t) \in \partial\Omega_T, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\partial I_{[-1,1]}(\cdot)$ denotes the sub-differential of the indicator function $I_{[-1,1]}(\cdot)$ and n denotes the unit exterior normal to $\partial\Omega$. The deep quench obstacle problem, (DQOP), constitutes the formal zero temperature ($\Theta \downarrow 0$) limit of the Cahn-Hilliard equation with degenerate mobility and with a logarithmic potential [10]:

$$\begin{cases} u_t = \nabla \cdot M(u)\nabla w, \quad w = \frac{\Theta}{2}\{\ln(1+u) - \ln(1-u)\} - u - \varepsilon^2 \triangle u, \quad (x,t) \in \Omega_T, \\ n \cdot \nabla u = n \cdot M(u)\nabla w = 0, \quad (x,t) \in \partial\Omega_T, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases}$$

Given a smoothly embedded curve $\Gamma = \Gamma(t) \subset \Omega$, the curve $\Gamma(t)$ is said to evolve by motion by surface diffusion [15] if, up to rescaling by constants,

$$\text{(SD)} \quad \begin{cases} V = -\kappa_{ss}, \quad (x(s,t),t) \in \Omega_T, \\ \Gamma(0) = \Gamma_0, \end{cases}$$

where s denotes an arc-length parametrization of $\Gamma(t)$, and V and κ denote, respectively, the normal velocity and the mean curvature of $\Gamma(t)$, defined in accordance with the exterior normal, n, relative to the arc-length parametrization, s.

Often it is convenient to express the deep quench obstacle problem with degenerate mobility somewhat informally as

$$\begin{cases} u_t = -\nabla \cdot M(u)\nabla(u+\varepsilon^2 \triangle u), \quad -1 \le u \le 1, \quad (x,t) \in \Omega_T, \\ n\cdot \nabla u = n \cdot M(u)\nabla(u+\varepsilon^2 \triangle u) = 0, \quad (x,t) \in \partial\Omega_T, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1}$$

As shown in [6], by considering (DQOP) on the slow time scale, $\tau = \varepsilon^2 t$, to leading order the $\varepsilon \downarrow 0$ limit of (DQOP) yields surface diffusion motion (SD) for $\Gamma = \Gamma(t)$, where $\Gamma(t)$ denotes the limit of the $O(\varepsilon)$ width interfaces which partition the composition $u(x,t)$ in the context of (DQOP) into two phases, $u = \pm 1$, in Ω. Off hand, $\Gamma(t)$ may contain one or more components.

With regard to existence, the following is implied by [2, Theorem 1.1]:

Theorem 1. *Let $\varepsilon > 0$ and $T > 0$, and let $\Omega \subset R^2$ be a bounded disk centered at the origin. Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product, and let $\langle\cdot,\cdot\rangle$ denote the $H^1(\Omega)$, $(H^1(\Omega))'$ duality pairing. Suppose that $u_0 \in \mathscr{K} := \{\eta \in H^1(\Omega)\,|\,|\eta| \le 1\}$. Then there exists a pair $\{u,w\}$, such that $u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))') \cap L^\infty(0,T;\mathscr{K})$, $w \in L^2(\Omega_T)$, with $w \in H^1_{loc}(\{M(u)>0\})$ for a.e. $t \in (0,T)$, and*

$$\begin{cases} \langle \frac{\partial u}{\partial t}, \eta\rangle + \int_{\{M(u)>0\}} \nabla w \cdot M(u) \nabla \eta \, dx = 0, \quad \forall \eta \in H^1(\Omega) \;\; a.e. \;\; t \in (0,T), \\ \varepsilon^2(\nabla u, \nabla\eta - \nabla u) - (u, \eta - u) \ge (w, \eta - u), \quad \forall \eta \in \mathscr{K} \;\; a.e. \;\; t \in (0,T), \\ u(x,0) = u_0, \quad x \in \Omega. \end{cases} \tag{2}$$

With regard to (SD), we follow the discussion in [22]. Let $\Gamma : R \to R^2$ be a regular smooth immersed plane curve, which is periodic and closed with period $P \in (0,\infty)$, so that in fact $\Gamma : S^1 \to R^2$. We shall assume throughout that Γ is parameterized by arc-length. With regard to local existence,

Theorem 2. *Suppose $\Gamma_0 : R \to R^2$ is a regular closed periodic curve parametrized by arc-length, of class $\mathscr{C}^2 \cap W^{2,2}$ with $||\kappa||_{L^2(\Gamma_0)} < \infty$. Then there exists $T \in (0,\infty]$ and a unique one-parameter family of immersions parametrised by arc-length, $\Gamma : R \times [0,T) \to R^2$, such that (i) $\Gamma(0,\cdot) = \Gamma_0$, (ii) $V = -\kappa_{ss}$, (iii) $\Gamma(\cdot,t)$ is of class $\mathscr{C}^\infty$ and periodic with period $|\Gamma(\cdot,t)|$, $\forall t \in (0,T)$, and (iv) T is maximal.*

The uniqueness mentioned in Theorem 2 is modulo rotations, translations, changes in orientation, in accordance with the natural group of invariances for geometric flows in general, and for (SD) in particular.

Though the theorem above is formulated for one regular immersed circular curve, clearly Theorem 2 readily generalizes to accommodate a finite union $\cup_{i\in I}\Gamma_i$, $I \subset \mathscr{N}$, of such curves, which suffices for the purpose of the discussion that follows.

Note that while the existence for (DQOP) is guaranteed by Theorem 1 for arbitrary $T > 0$, existence for (SD) is guaranteed by Theorem 2 only on some maximal interval. Indeed in the context of (SD) the maximal interval may well be finite for various reasons, for example, due to finite time self-intersection or curvature singularity formation. If we wish to consider and compare the steady states for both evolutions, the following theorem which prescribes conditions guaranteeing global existence for (SD) is helpful, and can be readily adapted for $\cup_{i\in I}\Gamma_i$, $I \subset \mathscr{N}$.

Theorem 3. *(see [22, Theorem 1.1]) Let $\Gamma_0 : S^1 \to R^2$ be a regular smooth immersed closed curve with finite enclosed signed area, $\mathscr{A}(\Gamma_0) > 0$ with $\int_{\Gamma_0} \kappa\, ds = 2\pi$. Then there exists a constant $K^* > 0$, such that if*

$$K_{\rm osc}(\Gamma_0) < K^* \ \textit{and}\ I(\Gamma_0) < \exp(K^*/(8\pi^2)),$$

where $K_{\rm osc}(\Gamma) = |\Gamma| \int_\Gamma (\kappa - \bar{\kappa})^2\, ds$, with $\bar{\kappa} = |\Gamma|^{-1} \int_\Gamma \kappa\, ds$, denotes the normalized oscillation of the curvature and $I = |\Gamma|^2/(4\pi\mathscr{A}(\Gamma))$ is the isoperimetric ratio, the (SD) evolution for $\Gamma : S^1 \times [0,T) \to R^2$, with Γ_0 as initial data, exists for all time and converges exponentially fast to a round circle with radius $\sqrt{\mathscr{A}(\Gamma_0)/\pi}$.

Remark 1. We remark that more regularity is required for the initial conditions in Theorems 2 and Theorem 3 than appears to be required for the initial conditions in Theorem 1. However in connecting the (SD) flow with the (DQOP) flow, the (SD) curve $\Gamma(t)$ corresponds rather naturally to the (DQOP) level set, $\{u(x,t) = 0 \,|\, (x,t) \in \Omega_T\}$, whose support lies within the set $\{(x,t) \in \Omega_T \,|\, \mathscr{M}(u(x,t)) > 0\}$ where the required regularity is guaranteed for $T > 0$.

With regard to prominent dynamical features for (DQOP) and (SD), notably

$$E(t) := \frac{1}{\varepsilon|\Omega|} \int_\Omega \{(1-u^2) + \varepsilon^2 |\nabla u|^2\}\, dx \quad \text{and} \quad \mathscr{L}(\mathrm{t}) := |\Gamma(\mathrm{t})|, \tag{3}$$

where $E(t)$ is a scaled free energy[1] for (DQOP), are monotonically non-increasing for (DQOP) and (SD), respectively. Furthermore,

$$\bar{u}(t) := \frac{1}{|\Omega|} \int_\Omega u(x,t)\, dx \quad \text{and} \quad \mathscr{A}(\mathrm{t}) = -\frac{1}{2} \int_0^{\mathscr{L}(\mathrm{t})} \Gamma(\mathrm{s},\mathrm{t})\mathrm{n}\, \mathrm{ds}, \tag{4}$$

where $\bar{u}(t)$, the mean mass of $u(x,t)$, and $\mathscr{A}(t)$, the signed area enclosed by $\Gamma(t)$, are invariant under the respective (DQOP) and (SD) evolutions. The Gamma limit ($\varepsilon \downarrow 0$) of the constrained mean mass ($\bar{u} = \bar{u}_0$) minimizers of $E(t)$ is well known [20, 12] to yield a curve Γ with prescribed enclosed signed area, $\mathscr{A}_0 = \mathscr{A}(0)$; this can be readily be shown to hold also for u constrained to lie in $\mathscr{K}$ [16, 20], with

$$\mathscr{A}_0 = \frac{1}{2}|\Omega|(1 - \bar{u}_0), \tag{5}$$

[1] Here $E(t)$ has been scaled so that typically as equilibrium is approached, $E(t) \propto \frac{L(t)}{|\Omega|}$, where $L(t)$ reflects the length of the interface of between the two phases following phase separation; see [2].

where $\bar{u}_0 := \bar{u}(0)$ in the context of our geometric and boundary assumptions. Notably, (DQOP) can be formulated as a conserved H^{-1} gradient flow with respect to the energy functional, $E(t)$, and (SD) can be formulated as an H^{-1} gradient flow with respect to $\mathscr{L}(t)$; see [21] and the discussion in Section 3. Moreover, the functionals

$$\mathrm{Ent}(t) := \frac{1}{|\Omega|}\int_\Omega \{(1-u)\ln(1-u) + (1+u)\ln(1+u)\}\, dx,$$

$$K_{\mathrm{osc}}(\Gamma(t)) = |\Gamma| \int_\Gamma (k-\bar{k})^2 ds,$$

are non-increasing along the respective (DQOP) and (SD) flows[2]. For (DQOP), for initial conditions $u_0 \in \mathscr{K}$ which correspond to a perturbation of $u_0 \equiv \bar{u}$, $\bar{u} \in (-1,1)$, the dynamics can be characterized in terms of an initial regime of linear instability and a long time coarsening regime, [13, 18, 17]. For further discussion, see [2].

2 Steady states

We assume both motions to be defined within $\Omega \subset R^2$, where Ω is a planar disc centered at the origin with radius R_0, where R_0 is $O(1)$. When considering time evolution, we set $\Omega_T = \Omega \times (0,T)$, where $0 < T \leq \infty$.

Note that in (DQOP), no flux and Neumann boundary conditions are implied. For simplicity, we shall henceforth assume, more specifically, that $u \equiv -1$ in a δ-neighborhood of $\partial\Omega$, with $\varepsilon \ll \delta \ll O(1)$. While this assumption implies the boundary conditions given in (DQOP), it somewhat limits the resultant dynamics and steady states. Similarly, in studying (SD), surface diffusion motions, within Ω, we consider evolving curves $\Gamma(t) \subset \Omega$ which, more specifically, satisfy $\Gamma(t) \subset \mathscr{B}_{R_0}(\delta)$, where $\mathscr{B}_{R_0}(\delta)$ refers to the open planar disc centered at the origin with radius R_0 which has a bounding annular neighborhood with width δ where $u \equiv -1$.

2.1 Steady states and energy minimizers for (SD)

In considering (SD) energy minimizers, by recalling (3) one may view the energy to be given by the length of the curve, $\mathscr{L}(t)$, to within scaling constants. We allow $\Gamma(t)$ to be comprised of a finite number of nonintersecting components, $\Gamma(t) = \cup_{i\in I}\Gamma_i(t)$, $I \subset \mathscr{N}$ with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. Accordingly, the evolution by (SD) of each component Γ_i, $i \in I$, may be prescribed as

[2] Here $\mathrm{Ent}(t)$ reflects the physical entropy of the system, while $E(t)$ is a (scaled) free energy.

$$V^i = -\kappa^i_{s_i s_i}, \tag{6}$$

where V^i and κ^i denote respectively, the normal velocity and the mean curvature of $\Gamma_i(t)$, and s_i denotes an arc-length parametrization of $\Gamma_i(t)$.

Within the context of these assumptions, it readily follows from (6) that steady states correspond of a finite disjoint union of circular curves. Since the area encompassed by curves evolving by surface diffusion is conserved under the evolution [22], it follows that $\sum_{i\in I} A(\Gamma_i(t))$ is a conserved quantity. Taking into account the isoperimetric inequality and (6), we may now conclude the following:

Theorem 4. *Under the assumptions outlined above, the set of minimum energy steady states for (SD) corresponds to the set of circular curves with radius $(A^*/\pi)^{1/2}$, with $A^* = \sum_{i\in I} A(\Gamma_i(0))$, which are located somewhere within $\mathscr{B}_{R_0}(\delta)$. Any finite union of disjoint circular curves such that $\sum_{i\in I} A(\Gamma_i(t)) = \sum_{i\in I} A(\Gamma_i(0))$ also corresponds to a steady state for (SD).*

2.2 Steady states and energy minimizers for (DQOP)

The energy minimizing steady states for (DQOP) correspond to the energy minimizers of $E(t)$ within the set $u \in \mathscr{K}$ and which satisfy $\bar{u} = \bar{u}_0$. Given the definition of the energy $E(t)$ and the geometry described at the beginning of this section, considerations of energy symmetrization and regularity [3], lead us to conclude:

Theorem 5. *Under the assumptions outlined above, the minimum energy steady states for (DQOP) are monotonically decreasing with respect to distance from the origin, modulo possible translations within $\mathscr{B}_{R_0}(\delta)$.*

Remark 2. Since the energy $E(t)$, as well as the problem formulated in (DQOP), are invariant under the transformation $u \to -u$, if we set u to equal $+1$ rather than -1 in the $\delta-$annular neighborhood of $\partial\Omega$, then the conclusion in Theorem 5 would have yielded that the minimum energy steady states are monotonically decreasing, modulo translation within $\mathscr{B}_{R_0}(\delta)$. Without the constraint that u equals ± 1 in a $\delta-$annular neighborhood of $\partial\Omega$, the energetics of possible additional steady states would need to be considered. Such additional steady states would include certain energy minimizing steady state solutions with "droplet like" ± 1 concentrations along the boundary, with lower energy than the axi-symmetric energy minimizing steady states with the same mean mass and their translates, discussed above. For simplicity, we focus here on a more limited set of steady states, which provide insight into the more general case.

It follows from Theorem 5 and Remark 2 that we should consider the set of axi-symmetric monotonically increasing steady states for (DQOP) and their translates that lie within $\mathscr{B}_{R_0}(\delta)$. Since we are looking for constrained mean mass minimizers, we should explore the set of the monotonically decreasing axi-symmetry steady states with prescribed mean mass, $\bar{u}$. This is undertaken in detail in the two subsections that follow. If a steady state $u \in \mathscr{K}$ equals -1 in a $\delta-$annular neighborhood of $\partial\Omega$ and increases (non-decreases) monotonically, then either (i) $u = 1$ is attained in a circular neighborhood of the origin or (ii) $u \in [-1,1)$ in Ω except perhaps at the origin. In case (i), the steady states contain unique monotonically decreasing annular transition region, see Fig. 1a and Section 2.3 for details. In case (ii), "dimple solutions" are possible, with $-1 < u \leq 1$ at the origin and with $-1 \leq u < 1$ elsewhere; this possibility is explored in Section 2.4. See Fig. 1b.

Before exploring the details of the radial solutions, let us recall that we wish to connect the solutions of (DQOP) with solutions of (SD). In considering the Gamma limit, the set of (DQOP) solutions are compared with (SD) solutions with similar mass. While solutions to (DQOP) depend on the parameter ε, the mean mass constraint is independent of ε. Let us recall (5). If $\bar{u}_0 = -1$, then trivially, $u \equiv -1$. If $\bar{u}_0 = +1$, then $u \equiv +1$ is implied by (5), which does not yield a possible solution due to the requirement that $u = -1$ in a $\delta-$annular neighborhood of $\partial\Omega$. If $\bar{u}_0 \in (-1,1)$ and $\bar{u}_0$ is not too close to -1, then the monotone axi-symmetric steady state solutions to (DQOP) can be expected to Gamma converge to -1 outside the origin in an annulus with width $R_0 - r_0$, and to converge to $+1$ in the disk centered at the origin with radius r_0, with the circular curve Γ with radius r_0 constituting an axi-symmetric steady state solution to (SD). Recalling (5), we get that if Ω is a disk with radius R_0, then

$$\bar{u}_0 = \frac{1}{|\Omega|}(2\mathscr{A}_0 - |\Omega|) = \frac{2r_0^2 - R_0^2}{R_0^2}. \tag{7}$$

Based on (7), for monotone steady state axisymmetric solutions of (DQOP) with $\bar{u} \in (-1,1)$, an ε independent **equivalent mean mass condition** is implied, namely

Proposition 1. ***Equivalent mean mass.*** *Let u be a monotonically decreasing axi-symmetric steady state solution to (DQOP) with $\bar{u} = \bar{u}_0 \in (-1,1)$ which lies within $\Omega = \mathscr{B}_{R_0}(\delta)$, where $\mathscr{B}_{R_0}(\delta)$ denotes the disk with radius R_0 centered at the origin which has a bounding annular neighborhood with width δ where $u \equiv -1$. Then u is radial, $u = u(r)$, and*

$$\int_0^{R_0} u(r) r\, dr = \frac{1}{2}\bar{u}R_0^2 = \frac{1}{2}(2r_0^2 - R_0^2), \tag{8}$$

for some $0 < r_0 < R_0 - \delta$.

Let us consider Fig. 1. Given $\mathscr{B}_{R_0}(\delta)$, it follows from (3) and (7) for $0 < \varepsilon \ll 1$, that if $\bar{u}_0$ is not too close to -1, then the solution can be expected to contain an annular transition region, $r_- < r < r_+$, with $0 < r_- < r_0 < r_+ < R_0 - \delta$ and

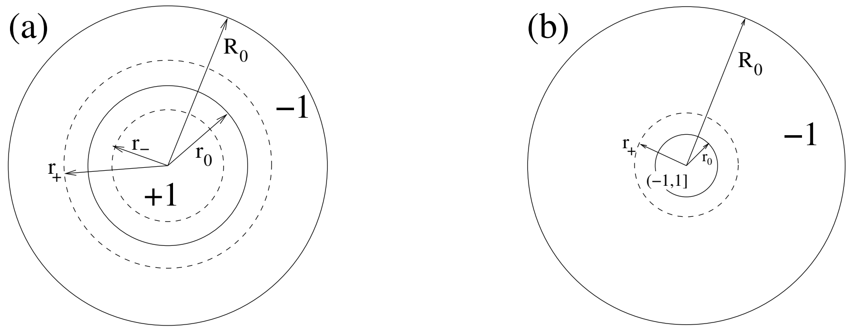

Fig. 1 a) Schematic portrayal of an annular solution: $u(r) \equiv 1$ for $0 \le r \le r_-$, $-1 < u(r) < 1$ for $r_- < r < r_+$, and $u(r) \equiv -1$ for $r_+ \le r < R_0$. b) Schematic portrayal of a dimple solution: $a := u(0) = \lim_{r\downarrow 0} u(r) \in (-1,1]$ with $-1 < u(r) \le a$ for $0 \le r < r_+$, and $u(r) \equiv -1$ for $r_+ \le r < R_0$.

$r_+ - r_- = O(\varepsilon)$, where $u(r) = -1$ for $r \in [r_+, R_0]$ and $u(r) = +1$ for $r \in [0, r_-]$; we will refer to such solutions as "annular" solutions. If, on the other hand, $\bar{u}_0$ is sufficient close to -1, then any possible (admissible) monotone radial solutions will have $u = -1$ in an annular region, $r_+ \le r \le R_0$, with $0 < r_0 < r_+ < R_0$, such that $u \in (-1,1)$ for $0 < r < r_+$ and $u(0) = \lim_{r\downarrow 0} u(r) \in (-1,1]$; we will refer to such solutions as "dimple" solutions. In either case, in accordance with (DQOP) and Theorem 1 (see [7]), $u \in C^1(\mathscr{B}_{R_0}(\delta))$ and outside of the annular or circular regions where $u \equiv \pm 1$, $u \in C^2$.

Accordingly, the annular solution should satisfy

$$
\begin{aligned}
&u + \lambda = -\varepsilon^2 (u_{rr} + \frac{1}{r} u_r), \qquad r_- < r < r_+, \\
&u(r_-) = 1\,, \quad u(r_+) = -1, \\
&u_r(r_-) = 0\,, \quad u_r(r_+) = 0,
\end{aligned}
\tag{9}
$$

for some r_-, r_+, with $0 < r_- < r_0 < r_+ < R_0$, and for some $\lambda \in R$. Given the boundary conditions in (9), λ can be considered here as a constant of integration for (6) with vanishing normal velocity. By testing the equation in (9) by $u(r)$, it is readily seen that λ can also be viewed as a mean mass conserving Lagrange multiplier for the free energy $E(t)$; see (11) as well as (14) in the sequel. Note that by (8)

$$
\bar{u} = \frac{(2r_0^2 - R_0^2)}{R_0^2} = \frac{2}{R_0^2} \int_0^{R_0} u(r) r\, dr, \tag{10}
$$

and from the prescribed structure of annular solutions

$$
\int_0^{R_0} u(r) r\, dr = \int_{r_-}^{r_+} u(r) r\, dr + \frac{1}{2}(r_+^2 + r_-^2 - R_0^2).
$$

Using (9),

$$\int_{r_-}^{r_+} u(r)r\,dr = -\int_{r_-}^{r_+} [(ru_r)_r - \lambda r]\,dr = \frac{\lambda}{2}(r_+^2 - r_-^2).$$

Hence the equivalent mean mass condition holds for (9) with (10) if and only if

$$\lambda = \frac{r_+^2 + r_-^2 - 2r_0^2}{r_+^2 - r_-^2}. \tag{11}$$

Similarly, the dimple solution should satisfy

$$\begin{aligned} &u + \lambda = -\varepsilon^2 (u_{rr} + \frac{1}{r} u_r), \qquad 0 < r < r_+, \\ &-1 < u(0) \le 1 \ , \quad u(r_+) = -1, \\ &u_r(0) = 0 \ , \quad u_r(r_+) = 0, \end{aligned} \tag{12}$$

for some r_+, with $0 < r_0 < r_+ < R_0$, and for some $\lambda \in R$. Again, by (8) in Proposition 1

$$\bar{u} = \frac{(2r_0^2 - R_0^2)}{R_0^2} = \frac{2}{R_0^2}\int_0^{R_0} u(r)r\,dr, \tag{13}$$

and from the prescribed structure of dimple solutions

$$\int_0^{R_0} u(r)r\,dr = \int_0^{r_+} u(r)r\,dr + \frac{1}{2}(r_+^2 - R_0^2).$$

Using (12),

$$\int_0^{r_+} u(r)r\,dr = -\int_0^{r_+} [(ru_r)_r - \lambda r]\,dr = \frac{\lambda}{2} r_+^2.$$

Hence the equivalent mean mass condition holds for (12) with (13) if and only if

$$\lambda = \frac{r_+^2 - 2r_0^2}{r_+^2}. \tag{14}$$

Before going into the details of the annular and dimple (DQOP) solutions in the next subsections, we pause to point out that it is possible to attain a large class of additional solutions by appropriately pasting together translates of dimple and annular solutions of various sizes, so long as the resulting construction lies in $\mathscr{K}$, for some $\bar{u} \in (-1,1)$. See, for example, the concentrically ringed solutions identified by X. Chen [5] as the asymptotic limit of solutions to the Cahn-Hilliard equation.

2.3 Annular solutions for (DQOP)

We now consider radial "annular" monotonically decreasing solutions in $\Omega = \mathscr{B}_{R_0}(\delta)$, containing a transition between the values ± 1, see Fig. 1a. More specifically we assume that $u(r) \equiv +1$ for $r \in [0, r_-]$, $-1 < u(r) < 1$ for $r \in (r_-, r_+)$, and

$u(r) \equiv -1$ for $r \in [r_+, R_0)$, where $r_\pm$ reflect the location of free boundaries with $0 < r_- < r_0 < r_+ < R_0 - \delta$. It is possible to seek non-monotone solutions, but these would have more energy and here we are looking for energy minimizing steady states. Generically we may assume that $r_0 = O(1)$, although, as we shall see, certain values of r_0 with $r_0 = O(\varepsilon)$ are also possible. We shall discuss the generic case first, and treat the general case afterwards.

As we saw in Section 2.2, $u(r)$ should satisfy

$$\begin{aligned} &u + \lambda = -\varepsilon^2 (u_{rr} + \frac{1}{r} u_r), \qquad r_- < r < r_+, \\ &u(r_-) = 1, \quad u(r_+) = -1, \\ &u_r(r_-) = 0, \quad u_r(r_+) = 0, \end{aligned}$$

where the equivalent mean mass condition holds if and only if

$$\bar{u} = \frac{(2r_0^2 - R_0^2)}{R_0^2}, \quad \lambda = \frac{r_+^2 + r_-^2 - 2r_0^2}{r_+^2 - r_-^2}, \quad r_0 = \Big[\frac{1+\bar{u}}{2}\Big]^{1/2} R_0, \tag{15}$$

for some $0 < r_0 < r_+$, see (9)–(11). Thus $\lambda = \lambda(r_-, r_+, r_0)$ and $r_0 = r_0(\bar{u}, R_0)$. We shall see that annular solutions exist for $-1 + O(\varepsilon^2) < \bar{u} < 1 - 4\delta/R_0 + O(\delta^2, \varepsilon)$ with $O(\varepsilon) < r_0 < R_0 - \delta + O(\varepsilon)$; some further specifics to follow.

The parameters, $r_\pm$, reflecting the location of the free boundaries, are to be determined. Thus, we have four boundary conditions, as well as the monotonicity and range constraints. In solving (9)–(11), we have four degrees of freedom, two from the second order ODE in (9) and two from the parameters $r_\pm$, with $\lambda = \lambda(r_-, r_+, r_0)$. Hence, one would expect the possible annular solutions to be uniquely determined by $\bar{u}$ (or, equivalently, by r_0 and R_0).

Setting

$$q = \frac{r}{\varepsilon}, \quad q_0 = \frac{r_0}{\varepsilon}, \quad q_\pm = \frac{r_\pm}{\varepsilon}, \quad Q_0 = \frac{R_0}{\varepsilon}, \quad \text{and} \quad v(q) := [u(r) + \lambda]|_{r=\varepsilon q}, \tag{16}$$

we get the following problem for $v(q)$,

$$\begin{aligned} &q v_{qq} + v_q + q v = 0, \quad 0 < q_- < q < q_+, \\ &v(q_-) = 1 + \lambda, \quad v(q_+) = -1 + \lambda, \\ &v_q(q_-) = 0, \quad v_q(q_+) = 0, \end{aligned} \tag{17}$$

with

$$\lambda = \frac{q_+^2 + q_-^2 - 2q_0^2}{q_+^2 - q_-^2}. \tag{18}$$

The equation in (17) is a Bessel equation of order zero, [8, 10.2.1], whose general solution ([8, 10.2(iii),10.6.3]) is

$$v(q) = c_1 J_0(q) + c_2 Y_0(q), \quad q > 0, \qquad c_1, c_2 \in R, \tag{19}$$

with

$$v_q(q) = -c_1 J_1(q) - c_2 Y_1(q), \quad q > 0.$$

The coefficients c_1, c_2, in the solution to (17), should depend on $q_\pm$ and λ, which in turn depend on $\bar{u}$ (or equivalently on r_0), as well as on the underlying parameters R_0 and ε.

Using the polar representation for Bessel functions, [8, 10.18.4, 10.18.6, 10.18.7], we get that

$$v(q) = c_1 J_0(q) + c_2 Y_0(q) = M_0(q)\big(c_1 \cos\theta_0(q) + c_2 \sin\theta_0(q)\big), \quad q > 0, \tag{20}$$

$$-v_q(q) = c_1 J_1(q) + c_2 Y_1(q) = M_1(q)\big(c_1 \cos\theta_1(q) + c_2 \sin\theta_1(q)\big), \quad q > 0, \tag{21}$$

where for $n = 0, 1$, $\theta_n(x) := \arctan(Y_n(x)/J_n(x))$ for $x > 0$, and

$$M_n(x) := \sqrt{J_n^2(x) + Y_n^2(x)} > 0, \quad x > 0,$$

since the Bessel functions J_n, Y_n do not vanish simultaneously.

Setting

$$\cos\varphi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}, \quad \sin\varphi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}, \quad \text{and} \quad A := \frac{1}{\sqrt{c_1^2 + c_2^2}}, \tag{22}$$

we can write (20),(21) as

$$v(q) = AM_0(q)\cos\big(\theta_0(q) - \varphi\big), \quad v_q(q) = -AM_1(q)\cos\big(\theta_1(q) - \varphi\big), \quad q > 0. \tag{23}$$

Since M_1 and A are positive, the boundary conditions $v_q(q_-) = v_q(q_+) = 0$ in (17) imply that

$$\cos\big(\theta_1(q_-) - \varphi\big) = \cos\big(\theta_1(q_+) - \varphi\big) = 0. \tag{24}$$

As we are seeking monotone solutions, $v(q)$ should be monotonically decreasing with $v_q < 0$ for $q_- < q < q_+$ by Sturmian theory. The positivity of M_1 and A now implies that $\cos\big(\theta_1(q) - \varphi\big) > 0$ for $q_- < q < q_+$, and thus that

$$\frac{-\pi}{2} < \theta_1(q) - \varphi < \frac{\pi}{2}, \quad 0 < q_- < q < q_+, \tag{25}$$

up to possible translations by $2k\pi$, $k \in Z$, which do not effect the solutions. The function $\theta_1(q)$ is monotonically increasing ([8, 10.18.18]); therefore

$$\theta_1(q_-) - \varphi = -\frac{\pi}{2}, \quad \theta_1(q_+) - \varphi = \frac{\pi}{2}, \quad \text{and} \quad \theta_1(q_+) - \theta_1(q_-) = \pi. \tag{26}$$

Since $\lim_{q \downarrow 0} \theta_1(q) = -\frac{\pi}{2}$ ([8, 10.18.3]), the monotonicity of $\theta_1(q)$ and (26) imply that

$$\theta_1(q_+) > \frac{\pi}{2}, \quad \theta_1(q_-) > -\frac{\pi}{2}, \quad \varphi > 0. \tag{27}$$

2.3.1 Annular solutions, the generic case

We remarked earlier that $-1+O(\varepsilon^2) < \bar{u} < 1-4\delta/R_0+O(\delta^2,\varepsilon)$ with $O(\varepsilon) < r_0 < R_0-\delta+O(\varepsilon)$. Thus, generically $r_0 = O(1)$. Considerations of scaling and energy minimization of the energy prescribed in (3) for $0 < \varepsilon \ll 1$ imply that transition widths between the phases for energy minimizing solutions scale as $O(\varepsilon)$, see e.g. [16, 20], and hence, $r_+ - r_- = O(\varepsilon)$ and r_-, r_+ are $O(1)$ when $r_0 = O(1)$. This will also be demonstrated directly in Section 2.3.2. Accordingly, the rescalings in (16) imply that

$$q_-, q_0, q_+ \,=\, O(\varepsilon^{-1}), \quad q_+ - q_- = O(1), \quad \varepsilon q_0 = O(1) \tag{28}$$

in the generic case. Throughout this subsection, we assume that (28) holds.

For large values of x (see [8, 10.18.18]),

$$\theta_1(x) = x - \frac{3\pi}{4} + \frac{3}{8x} + O\left(\frac{1}{x^3}\right), \quad x \gg 1. \tag{29}$$

Hence, by (26), for large values of q_-, q_+,

$$q_- - \frac{3\pi}{4} + \frac{3}{8q_-} + O\left(\frac{1}{q_0^3}\right) - \varphi = -\frac{\pi}{2}, \quad q_+ - \frac{3\pi}{4} + \frac{3}{8q_+} + O\left(\frac{1}{q_0^3}\right) - \varphi = \frac{\pi}{2}, \tag{30}$$

and thus

$$q_+ - q_- = \pi + \frac{3}{8}\left(\frac{1}{q_-} - \frac{1}{q_+}\right) + O\left(\frac{1}{q_0^3}\right). \tag{31}$$

The boundary conditions $v(q_-) = 1+\lambda$ and $v(q_+) = -1+\lambda$ in (17) imply that

$$AM_0(q_-)\cos\big(\theta_0(q_-)-\varphi\big) = 1+\lambda, \quad AM_0(q_+)\cos\big(\theta_0(q_+)-\varphi\big) = -1+\lambda. \tag{32}$$

Subtracting the equations in (32),

$$M_0(q_-)\cos\big(\theta_0(q_-)-\varphi\big) - M_0(q_+)\cos\big(\theta_0(q_+)-\varphi\big) = \frac{2}{A}. \tag{33}$$

For large values of x (see [8, 10.18.18]),

$$\theta_0(x) = x - \frac{\pi}{4} - \frac{1}{8x} + O\left(\frac{1}{x^3}\right), \quad x \gg 1.$$

Hence using (30)

$$\theta_0(q_-) - \varphi = -\frac{1}{2q_-} + O\left(\frac{1}{q_0^3}\right), \quad \theta_0(q_+) - \varphi = \pi - \frac{1}{2q_+} + O\left(\frac{1}{q_0^3}\right),$$

and therefore

$$\cos(\theta_0(q_-)-\varphi) = 1-\frac{1}{8q_-^2}+O\left(\frac{1}{q_0^3}\right), \quad \cos(\theta_0(q_+)-\varphi) = -1+\frac{1}{8q_+^2}+O\left(\frac{1}{q_0^3}\right). \tag{34}$$

For large values of x (see [8, 10.18.17]),

$$M_0(x) = \sqrt{\frac{2}{\pi x}}+O\left(\frac{1}{x^{5/2}}\right), \quad x \gg 1. \tag{35}$$

Using (34), (35) in (32),

$$\sqrt{\frac{2}{\pi q_-}}\cdot\left(1-\frac{1}{8q_-^2}\right)-\sqrt{\frac{2}{\pi q_+}}\cdot\left(-1+\frac{1}{8q_+^2}\right)+O\left(\frac{1}{q_0^{5/2}}\right) = \frac{2}{A},$$

which implies that

$$A = \frac{\sqrt{2\pi}}{\frac{1}{\sqrt{q_-}}+\frac{1}{\sqrt{q_+}}}+O\left(\frac{1}{q_0^{3/2}}\right). \tag{36}$$

Returning to (32) and summing the two equations,

$$AM_0(q_-)\cos\big(\theta_0(q_-)-\varphi\big)+AM_0(q_+)\cos\big(\theta_0(q_+)-\varphi\big) = 2\lambda,$$

and then using the approximations in (34), (35),

$$\left(\frac{\sqrt{2\pi}}{\frac{1}{\sqrt{q_-}}+\frac{1}{\sqrt{q_+}}}+O\left(\frac{1}{q_0^{3/2}}\right)\right)\left(\sqrt{\frac{2}{\pi q_-}}-\sqrt{\frac{2}{\pi q_+}}+O\left(\frac{1}{q_0^{5/2}}\right)\right) = 2\lambda,$$

which implies that

$$\lambda = \frac{\sqrt{q_+}-\sqrt{q_-}}{\sqrt{q_+}+\sqrt{q_-}}+O\left(\frac{1}{q_0^2}\right). \tag{37}$$

Using now (31),

$$\lambda = \frac{\pi+O\left(\frac{1}{q_0}\right)}{(\sqrt{q_-+\pi}+\sqrt{q_-})^2} = O\left(\frac{1}{q_0}\right),$$

from which we get that

$$q_- = \frac{\pi(1-2\lambda)}{4\lambda}+O\left(\frac{1}{q_0}\right), \quad q_+ = \frac{\pi(1+2\lambda)}{4\lambda}+O\left(\frac{1}{q_0}\right), \tag{38}$$

which implies that $q_\pm = O(q_0)$ and $q_+ - q_- = O(1)$, in accordance with (28).

Using (38) and (16) in the expression for λ given in (15) and noting that (37) implies that $\lambda > 0$, then solving for λ, we obtain that

$$\lambda = \frac{\pi}{4q_0}+O\left(\frac{1}{q_0^2}\right). \tag{39}$$

Substituting the above expression for λ into (38) we get that

$$q_- = q_0 - \frac{\pi}{2} + O\left(\frac{1}{q_0}\right) \quad q_+ = q_0 + \frac{\pi}{2} + O\left(\frac{1}{q_0}\right).$$

In order to obtain an approximate solution $v(q)$ to (17), and subsequently to obtain an approximate solution $u(r) = v(r/\varepsilon) - \lambda$ to (9), based, say on the expression in (19), in the generic case, it remains to identify approximations for the coefficients, c_1, c_2.

The boundary conditions $v(q_+) = -1 + \lambda$ and $v_q(q_+) = 0$ in (17), imply that

$$c_1 = \frac{(-1+\lambda)Y_1(q_+)}{J_0(q_+)Y_1(q_+) + J_1(q_+)Y_0(q_+)}, \quad c_2 = \frac{(-1+\lambda)Y_0(q_+)}{J_0(q_+)Y_1(q_+) + J_1(q_+)Y_0(q_+)}. \tag{40}$$

We know (see [8, 10.17.3, 10.17.4]) that

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right), \quad x \gg 1,$$
$$Y_0(x) = \sqrt{\frac{2}{\pi x}}\left(\sin\left(x - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right), \quad x \gg 1,$$

and that

$$J_1(x) = \sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{3\pi}{4}\right) + O\left(\frac{1}{x}\right)\right), \quad x \gg 1,$$
$$Y_1(x) = \sqrt{\frac{2}{\pi x}}\left(\sin\left(x - \frac{3\pi}{4}\right) + O\left(\frac{1}{x}\right)\right), \quad x \gg 1.$$

Using the approximations above in (40),

$$c_1 = -\sqrt{\frac{q_0\pi}{2}}\,\frac{[\sin(q_0 - \pi/4)]}{\sin(2q_0)} + O\left(\frac{1}{q_0}\right), \quad c_2 = -\sqrt{\frac{q_0\pi}{2}}\,\frac{[\cos(q_0 - \pi/4)]}{\sin(2q_0)} + O\left(\frac{1}{q_0}\right), \tag{41}$$

and then returning to (19), we obtain that

$$v(q) = -\sqrt{\frac{q_0}{q}}\,\frac{\cos(q+q_0)}{\sin(2q_0)} + O\left(\frac{1}{q_0}\right), \quad q_- < q < q_+ \ (q_0 - \frac{\pi}{2} < q < q_0 + \frac{\pi}{2}\ ). \tag{42}$$

Recalling (16),

$$u(r) = -\sqrt{\frac{r_0}{r}}\,\frac{\cos((r+r_0)/\varepsilon)}{\sin(2r_0/\varepsilon)} + O\left(\frac{\varepsilon}{r_0}\right), \quad r_- < r < r_+. \tag{43}$$

2.3.2 Existence and Uniqueness

In this section, we prove

Theorem 6. *Given* ε, δ, *and* R_0, $0 < \varepsilon \ll \delta \ll 1$, $R_0 = O(1)$. *There exists a unique solution to (9), (11) in* $\mathscr{B}_{R_0}(\delta)$ *for* $r_0 \in (r_0^{\inf}, r_0^{\sup}]$, *where* $r_0^{\inf} = \bar{q}/\varepsilon$ *with* $\bar{q} := \theta_1^{-1}(\pi/2)$ *and* $r_0^{\sup} = R_0 - \delta + O(\varepsilon)$.

Remark 3. In Theorem 6, $r_0^{\inf}$ corresponds to the largest value of r_0 for which dimple solutions exist, to be discussed in detail in Section 2.4; for the annular solutions, $r_- \downarrow 0$ as $r_0 \downarrow r_0^{\inf}$. The upper limit, $r_0^{\sup}$, corresponds to the largest possible value of r_0, given the $\delta-$width annulus where $u \equiv -1$ within $\mathscr{B}_{R_0}(\delta)$, with $r_+ \uparrow R_0 - \delta$ as $r_0 \uparrow r_0^{\sup}$.

Proof. It is convenient to prove Theorem 6 by using the rescalings defined in (16), and proving unique existence for the equivalent problem prescribed in (17), (18) for $q_0 \in (q_0^{\inf}, q_0^{\sup}]$, where $q_0^{\inf} = \bar{q} = \varepsilon r_0^{\inf}$, $q_0^{\sup} = \varepsilon r_0^{\sup}$. In Proposition 2 below, unique existence is demonstrated for $q_0 \in (\bar{q}, \infty)$, and then the implied ranges indicated in Theorem 6 follows by returning to the original scaling and imposing $r_0^{\sup}$ the upper limit, with $r_0^{\sup}$ corresponding to $r_+^{\sup} = R_0 - \delta$. In the course of the proof of Proposition 2, we shall see that $r_0^{\sup} - r_+^{\sup} = O(\varepsilon)$, which implies that $r_0^{\sup} = R_0 - \delta + O(\varepsilon)$.

Proposition 2. *There exists a unique monotone solution to (17)-(18) for every* $q_0 \in (\bar{q}, \infty)$, *where* $\bar{q} := \theta_1^{-1}(\pi/2)$ *corresponds to the first positive root of* $J_1(q)$.

Proof. The proof relies on introducing a functional, $D = D(t)$, defined below, which allows us to focus first on existence and then on uniqueness. Let us recall that the general solution to the equation in (17) may be written as in (23), namely as

$$v(q) = AM_0(q)\cos\big(\theta_0(q) - \varphi\big), \quad v_q(q) = -AM_1(q)\cos\big(\theta_1(q) - \varphi\big), \quad q \geq 0,$$

where $A > 0$, $M_0(q)$, $M_1(q) > 0$, for $q \geq 0$. The boundary conditions $v_q(q_-) = v_q(q_+) = 0$ in (17), together with the monotonicity of $\theta_1(x)$ with $\lim_{x\downarrow 0}\theta_1(x) = -\pi/2$, imply that

$$\varphi > 0, \quad \frac{-\pi}{2} < \theta_1(q) - \varphi < \frac{\pi}{2}, \quad 0 < q_- < q < q_+, \quad \theta_1(q_\pm) - \varphi = \pm\frac{\pi}{2}, \quad (44)$$

up to possible translations by $2k\pi$, $k \in Z$, see (25)–(27).

The general solution contains two parameters, A and φ. As we have already accommodated the boundary conditions $v_q(q_-) = v_q(q_+) = 0$, finding a solution to (17), (18) entails identifying q_- and q_+ as functions of A and φ so as to satisfy the two remaining boundary conditions in (17). In view of (44), it is convenient to work with the parameters A and t, rather than with the parameters A and φ, where

$$t = \theta_1(q_0) - \varphi, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}, \tag{45}$$

up to possible translation by $2k\pi$, $k \in Z$. From (44), (45), we get that

$$\varphi(t) = \theta_1(q_0) - t, \quad \theta_1(q_\pm(t)) = \theta_1(q_0) - t \pm \frac{\pi}{2}, \quad t \in [-\pi/2, \pi/2]. \tag{46}$$

Since $\theta_1(x)$ is monotonically increasing and continuously differentiable for $x > 0$ ([8, 10.18.8]), $\theta_1(x)$ has an inverse function which is also monotonically increasing and continuously differentiable, and

$$q_\pm(t) = \theta_1^{-1}\left(\theta_1(q_0) - t \pm \frac{\pi}{2}\right). \tag{47}$$

Thus φ, q_-, q_+, and well as λ (see (18)), can be viewed as continuously differentiable functions of t, since $\theta_1(q_0) - t - \pi/2 > -\pi/2$ for $t \in [-\pi/2, \pi/2]$, for $q_0 > \theta_1^{-1}(\pi/2) = \bar{q}$.

From (23) and the boundary conditions, $v(q_-) = 1 + \lambda$ and $v(q_+) = -1 + \lambda$,

$$AM_0(q_-)\cos\big(\theta_0(q_-) - \varphi\big) = 1 + \lambda, \quad AM_0(q_+)\cos\big(\theta_0(q_+) - \varphi\big) = -1 + \lambda. \tag{48}$$

As φ, q_-, q_+, and λ are prescribed in terms of t, we get two descriptions for A in terms of t from (48) as long as $\cos\big(\theta_0(q_-) - \varphi\big)$ and $\cos\big(\theta_0(q_+) - \varphi\big)$ do not vanish, since $M_0(x) > 0$ for $x > 0$. Moreover, the two descriptions must be equal for some value of $t \in [-\pi/2, \pi/2]$, if we are to attain a solution to (17), (18). First we prove

Lemma 1. $\cos\big(\theta_0(q_-) - \varphi\big) > 0$, *and* $\cos\big(\theta_0(q_+) - \varphi\big) < 0$.

Proof. First we note that for $x \ge 0$,

$$\begin{aligned} J_0(x) &= M_0(x)\cos\theta_0(x), \quad J_0'(x) = N_0(x)\cos\phi_0(x) = -M_1(x)\cos\theta_1(x), \\ Y_0(x) &= M_0(x)\sin\theta_0(x), \quad Y_0'(x) = N_0(x)\sin\phi_0(x) = -M_1(x)\sin\theta_1(x). \end{aligned} \tag{49}$$

Using (49), the formula [8, 10.18.12], $M_0(x)N_0(x)\sin(\phi_0(x) - \theta_0(x)) = \frac{2}{\pi x}, \quad x > 0$, and a little trigonometry,

$$M_0(x)M_1(x)\sin(\theta_0(x) - \theta_1(x)) = \frac{2}{\pi x}, \quad x > 0. \tag{50}$$

Since $M_0(x), M_1(x) > 0$ for $x > 0$, (50) implies that $\sin(\theta_0(x) - \theta_1(x)) > 0, \quad x > 0$. Thus

$$0 < \theta_0(x) - \theta_1(x) < \pi, \quad x > 0,$$

up to possible translation by $2k\pi$, $k \in Z$. Therefore,

$$\theta_1(x) < \theta_0(x) < \theta_1(x) + \pi, \quad x > 0. \tag{51}$$

Setting $x = q_-$ in (51), we obtain that $\theta_1(q_-) - \varphi < \theta_0(q_-) - \varphi < \theta_1(q_-) - \varphi + \pi$. Then using (44),

$$-\frac{\pi}{2} < \theta_0(q_-) - \varphi < \frac{\pi}{2}$$

and therefore $\cos\big(\theta_0(q_-) - \varphi\big) > 0$. Similarly, we obtain from (51) and (44) that

$$\frac{\pi}{2} < \theta_0(q_+) - \varphi < \frac{3\pi}{2}$$

and therefore $\cos\big(\theta_0(q_+) - \varphi\big) < 0$. □

Given Lemma 1, we now obtain two expressions for A, namely,

$$A = \frac{1+\lambda}{M_0(q_-)\cos\big(\theta_0(q_-) - \varphi\big)} = \frac{-1+\lambda}{M_0(q_+)\cos\big(\theta_0(q_+) - \varphi\big)}. \tag{52}$$

Using the definition of λ given in (11), and noting that $\theta_1(q_+) - \theta_1(q_-) = \pi$ implies that $q_+ - q_- > 0$, we obtain from (52) that

$$\frac{q_+^2 - q_0^2}{M_0(q_-)\cos\big(\theta_0(q_-) - \varphi\big)} = \frac{q_-^2 - q_0^2}{M_0(q_+)\cos\big(\theta_0(q_+) - \varphi\big)},$$

where all the terms are continuously differentiable functions of t. It remains to verify that the above equation uniquely defines t. Cross-multiplying, we obtain now that roots of $D(t) = 0$, with $t \in [-\pi/2, \pi/2]$, correspond to solutions of (17), (18), where

$$D(t) := \mathscr{D}(q_+(t)) - \mathscr{D}(q_-(t)), \quad \mathscr{D}(q) := (q^2 - q_0^2)M_0(q)\cos\big(\theta_0(q) - \varphi\big), \tag{53}$$

is continuously differentiable for $t \in [-\pi/2, \pi/2]$. Existence now follows from the following lemma.

Lemma 2. $D(-\pi/2) < 0, \quad D(\pi/2) > 0.$

Proof. Let us first consider $D(t)$ with $t = -\pi/2$. From (46), $\theta_1(q_-(-\pi/2)) = \theta_1(q_0) > \pi/2$ for $q_0 > \bar{q}$. Recalling the monotonicity of $\theta_1(x)$ for $x > 0$, and using (47),

$$q_-(-\pi/2) = q_0, \quad q_+(-\pi/2) = \theta_1^{-1}(\theta_1(q_0) + \pi) > q_0. \tag{54}$$

Similarly for $D(t)$ with $t = \pi/2$, we obtain that $\theta_1(q_+(\pi/2)) = \theta_1(q_0) > \pi/2$, and hence

$$q_-(\pi/2) = \theta_1^{-1}(\theta_1(q_0) - \pi) < q_0, \quad q_+(\pi/2) = q_0. \tag{55}$$

Let us recall Lemma 1 and the positivity of $M_0(x)$ for $x > 0$. Then using (54) and the definition of $D(t)$ in (53),

$$D(-\pi/2) = (q_+(-\pi/2)^2 - q_0^2)M_0(q_+(-\pi/2))\cos\big(\theta_0(q_+(-\pi/2)) - \varphi(-\pi/2)\big) < 0.$$

Similarly, using (55) in (53), we obtain that $D(\pi/2) > 0$. □

Uniqueness now follows by proving the following

Lemma 3. $D'(t) > 0$, $t \in (-\pi/2, \pi/2)$.

Proof. Our proof is based on calculating $D'(t)$ where $D(t)$, given in (53). We begin by calculating $\varphi'(t)$, $q'_\pm(t)$, $\theta'_0(q_\pm(t))q'_\pm(t)$, and $M'_0(q_\pm(t))q'_\pm(t)$.

We recall the formulas from [8, 10.18.8],

$$M_0^2(x)\theta'_0(x) = \frac{2}{\pi x}, \quad M_1^2(x)\theta'_1(x) = \frac{2}{\pi x}, \quad x > 0. \tag{56}$$

Since $M_0(x)$ and $M_1(x) > 0$ are positive, (56) implies that

$$\theta'_0(x) = \frac{2}{\pi x}\frac{1}{\big(M_0(x)\big)^2}, \quad \theta'_1(x) = \frac{2}{\pi x}\frac{1}{\big(M_1(x)\big)^2} \quad x > 0. \tag{57}$$

It follows from (45) that

$$\varphi'(t) = -1. \tag{58}$$

From (46) and the invertibility of $\theta_1(x)$ for $x > 0$,

$$q_\pm = q_\pm(t) = \theta_1^{-1}\left(\theta_1(q_0) - t \pm \frac{\pi}{2}\right), \tag{59}$$

and therefore $\theta'_1(q_\pm(t))q'_\pm(t) = -1$. Recalling (57), we obtain that

$$q'_\pm(t) = -\frac{\pi q_\pm}{2}\left(M_1^2(q_\pm)\right), \quad \theta'_0(q_\pm(t))q'_\pm(t) = -\left(\frac{M_1^2(q_\pm)}{M_0^2(q_\pm)}\right). \tag{60}$$

Let us consider now the formula [8, 10.18.11], $M_0\theta'_0/M'_0 = \tan(\phi_0 - \theta_0)$. Recalling (49), we see that $\tan\phi_0 = \tan\theta_1$, and

$$\frac{\sin\phi_0}{\cos\phi_0} = \frac{N_0\sin\phi_0}{N_0\cos\phi_0} = \frac{-M_1\sin\theta_1}{-M_1\cos\theta_1}, \tag{61}$$

and therefore using (60),

$$M'_0(q_\pm) = M_0(q_\pm)\theta'_0(q_\pm)\cot(\theta_1(q_\pm) - \theta_0(q_\pm)),$$

$$M'_0(q_\pm)q'_\pm = \frac{M_1^2(q_\pm)}{M_0(q_\pm)}\cot(\theta_0(q_\pm) - \theta_1(q_\pm)).$$

Since $\theta_0(q_+) - \theta_1(q_+) = \big(\theta_0(q_+) - \varphi\big) - \big(\theta_1(q_+) - \varphi\big) = \big(\theta_0(q_+) - \varphi\big) - \frac{\pi}{2}$, we get that

$$\cot\big(\theta_0(q_+) - \theta_1(q_+)\big) = -\tan\big(\theta_0(q_+) - \varphi\big), \tag{62}$$

and similarly

$$\cot\big(\theta_0(q_-) - \theta_1(q_-)\big) = -\tan\big(\theta_0(q_-) - \varphi\big). \tag{63}$$

Thus

$$M_0'(q_\pm(t))q_\pm'(t) = -\frac{M_1^2(q_\pm)}{M_0(q_\pm)}\tan\big(\theta_0(q_\pm)-\varphi\big). \tag{64}$$

Recalling (53), let us differentiate $\mathscr{D}(q_\pm(t))$. Using the results above,

$$\begin{aligned}\frac{d}{dt}\mathscr{D}(q_\pm(t) &= (q_\pm^2-q_0^2)M_0(q_\pm)\cos(\theta_0(q_\pm)-\varphi)\\ &= -\pi q_\pm^2 M_0(q_\pm)M_1^2(q_\pm)\cos(\theta_0(q_\pm)-\varphi)-(q^2-q_0^2)M_0(q)\sin(\theta_0(q)-\varphi).\end{aligned} \tag{65}$$

We note that $\sin(\theta_0(q)-\varphi)=\tan(\theta_0(q)-\varphi)\cos(\theta_0(q)-\varphi)$, and using (61)-(63)) we get that $\tan(\theta_0(q)-\varphi)=+\cot(\phi_0(q)-\theta_0(q))$. By formula [8, 10.18.11], $\tan(\phi_0-\theta_0)=2/(\pi x M_0 M_0')$ for $x>0$, and hence

$$\tan(\theta_0(q)-\varphi)=\cot(\phi_0(q)-\theta_0(q))=\tfrac{1}{2}\pi q M_0 M_0' = \tfrac{1}{4}\pi q\left(M_0^2\right)'. \tag{66}$$

Therefore

$$\frac{d}{dt}\mathscr{D}(q_\pm(t)) = -\left(q_\pm M_1^2(q_\pm)+\tfrac{1}{4}(q_\pm^2-q_0^2)\left(M_0^2(q_\pm)\right)'\right)\pi q_\pm M_0(q_\pm)\cos(\theta_0(q_\pm)-\varphi), \tag{67}$$

where $q_\pm = q_\pm(t)$, $\varphi=\varphi(t)$. Let us, then, consider the expression

$$P(x) := x\big(M_1(x)\big)^2+\tfrac{1}{4}(x^2-q_0^2)\left(M_0(x)^2\right)', \quad x>0. \tag{68}$$

Since $N_0^2(x)=M_1^2(x)$ for $x\geq 0$ by (49), the formula [8, 10.18.10], $x^2M_0(x)M_0'(x)+x^2N_0(x)N_0'(x)+x\big(N_0(x)\big)^2=0$, $x\geq 0$, implies that

$$x\big(M_1(x)\big)^2 = -\tfrac{1}{2}x^2(M_0^2(x))'-\tfrac{1}{2}x^2(M_1^2(x))', \quad x\geq 0.$$

Therefore,

$$P(x) = -\tfrac{1}{4}x^2\left(M_0^2(x)\right)'-\tfrac{1}{4}q_0^2\left(M_0^2(x)\right)'-\tfrac{1}{2}x^2\left(M_1^2(x)\right)', \quad x>0.$$

The claim below implies that $P(x)>0$ for $x>0$.

Claim. $\left(M_0^2(x)\right)'<0$ and $\left(M_1^2(x)\right)'<0$, for $x>0$.

Proof. To prove the claim, we use the Nicholson's Integral Representation (see [8, 10.9.30]), which implies that for $x>0$,

$$M_0^2(x)=\frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\,dt, \quad M_1^2(x)=\frac{8}{\pi^2}\int_0^\infty \cosh(2t)\,K_0(2x\sinh t)\,dt, \tag{69}$$

and using the formula (see [8, 10.29.3] that

$$K_0'(x) = -K_1(x), \quad x>0, \tag{70}$$

where in (69), (70), $K_0(x)$ and $K_1(x)$ denote the second standard solutions to the Modified Bessel equation, with $\nu = 0$, 1, respectively.

By considering the asymptotic behavior of $K_0(x)$ and $K_1(x)$ for $0 < x \ll 1$ [8, 10.30.2,10.30.3] and $x \gg 1$ [8, 10.25.3], it readily follows that the formal differentiation of the (convergent) representations for $M_0(x)$ and $M_1(x)$ given in (69) is justified,

$$\big(M_0^2(x)\big)' = -\frac{8}{\pi^2}\int_0^\infty K_1(2x\sinh t)\cdot 2\sinh t\,dt, \quad x > 0,$$

$$\big(M_1^2(x)\big)' = -\frac{8}{\pi^2}\int_0^\infty \cosh(2t)\,K_1(2x\sinh t)\cdot 2\sinh t\,dt, \quad x > 0,$$

as the integrals above are convergent uniformly in x, for $x > 0$. As the functions $\sinh(x)$, $\cosh(x)$ and $K_1(x)$ are strictly positive for $x > 0$, the claim follows. □

To complete the proof of Lemma 3, let us recall that $M_0(x) > 0$ for $x > 0$, and $\cos\big(\theta_0(q_+) - \varphi\big) < 0$ and $\cos\big(\theta_0(q_-) - \varphi\big) > 0$ by Lemma 1. Hence the above claim implies that $\pm\frac{d}{dt}\mathscr{D}(q_\pm(t)) > 0$, and therefor $D'(t) > 0$ in accordance with the definition of $D(t)$ in (53). □

This completes the proof of Proposition 2. □

To return now and complete the proof of Theorem 6, let $q_+^{\sup} = (R_0 - \delta)/\varepsilon$ and let us consider the corresponding value of $q_0^{\sup}$. Since $q_+^{\sup} = \theta_1^{-1}(\theta_1(q_0^{\sup} - t + \pi/2))$ for some $t \in [-\pi/1, \pi/2]$ by (47), we obtain that $q_+^{\sup} = q_0^{\sup} + O(1)$ and hence $r_0^{\sup} = r_+^{\sup} + O(\varepsilon)$ as claimed earlier. □

2.4 Dimple solutions for (DQOP)

We now consider radial "dimple" solution in $\Omega = \mathscr{B}_{R_0}(\delta)$, with $-1 < u(r) < 1$ for $r \in (0, r_+)$ and $u(r) \equiv -1$ for $r \in [r_+, R_0)$, where r_+ reflects a free boundary with $0 < r_+ < R_0 - \delta$. As we saw in Section 2.2, $u(r)$ should satisfy

$$\begin{aligned} &u + \lambda = -\varepsilon^2(u_{rr} + \frac{1}{r}u_r), \quad 0 < r < r_+, \\ &u(0) \in (-1, 1]\,, \quad u(r_+) = -1, \\ &u_r(0) = 0\,, \quad u_r(r_+) = 0, \end{aligned}$$

and the equivalent mean mass condition implies that

$$\bar{u} = \frac{(2r_0^2 - R_0^2)}{R_0^2}, \quad \lambda = \frac{r_+^2 - 2r_0^2}{r_+^2}, \quad r_0 = \Big[\frac{1+\bar{u}}{2}\Big]^{1/2} R_0, \tag{71}$$

for some $0 < r_0 < r_+$, see (12)–(14). Thus $\lambda = \lambda(r_+, r_0)$ and $r_0 = r_0(\bar{u}, R_0)$. We shall see that for $0 < \varepsilon \ll 1$, nontrivial radial dimple solutions exist for all $-1 <$

$\bar{u} = -1 + O(\varepsilon)$ and $0 < r_0 = O(\varepsilon)$ sufficiently small; in particular, as $\varepsilon \downarrow 0$, $\bar{u} \downarrow -1$, $r_0 \downarrow 0$, and $u(r;\varepsilon) \downarrow -1$, $0 < r < R_0$.

The parameter r_+, reflecting the location of the free boundary, is to be determined. The value $u(0) \in (-1,1]$ is as an additional free parameter to be determined. Thus, we have three boundary conditions and the equivalent mean mass constraint, $\bar{u} = (2r_0^2 - R_0^2)/R_0^2$, as well as the range constraint on $u(0)$. In solving (12)–(14), we have four degrees of freedom, counting the parameters $u(0)$ and r_+, with $\lambda = \lambda(r_+, r_0)$. Hence, from the "count" of the parameters, we would expect the possible dimple solutions to be uniquely determined by $\bar{u}$ (or, equivalently, by r_0).

Setting

$$q = \frac{r}{\varepsilon}, \quad q_0 = \frac{r_0}{\varepsilon}, \quad q_+ = \frac{r_+}{\varepsilon}, \quad Q_0 = \frac{R_0}{\varepsilon}, \quad \text{and} \quad v(q) := [u(r) + \lambda]|_{r=\varepsilon q}, \tag{72}$$

we get the following problem for $v(q)$,

$$\begin{aligned} &q v_{qq} + v_q + qv = 0, \qquad 0 < q < q_+, \\ &v(0) = u(0) + \lambda \in (\lambda - 1, \lambda + 1], \quad v(q_+) = -1 + \lambda, \\ &v_q(0) = 0, \quad v_q(q_+) = 0, \end{aligned} \tag{73}$$

where $\lambda = q_+^2 - 2q_0^2/q_+^2$. The equation in (73) is a Bessel's equation of order zero, whose general solution is $v(q) = c_1 J_0(q) + c_2 Y_0(q)$, $c_1, c_2 \in R$, and hence $v_q(q) = -c_1 J_1(q) - c_2 Y_1(q))$. Since $J_0(0) = 1$, $J_1(0) = 0$, and $Y_1(0) \neq 0$, the boundary conditions at $q = 0$ imply that $c_1 = u(0) + \lambda$, $c_2 = 0$. Thus

$$v(q) = (u(0) + \lambda) J_0(q), \quad v_q(q) = -(u(0) + \lambda) J_1(q). \tag{74}$$

The values of J_0 at its sequential minima are increasing [8, 10.3,10.21,10.18], and $v_q(q_+) = 0$ with $v(q_+) = -1 + \lambda$, which corresponds to $u_r(r_+) = 0$, $u(r_+) = -1$. Hence the range constraint, $-1 \leq u(r) \leq 1$, implies that $q_+ = \bar{q}$, where $\bar{q}$ corresponds to the (unique) first positive zero of the function $J_1(q)$ [8, 10.21]. Thus [1, Tables 9.1, 9.5],

$$q_+ = \bar{q} \approx 3.8 \quad \text{with} \quad J_0(q_+) = J_0(\bar{q}) \approx -0.4. \tag{75}$$

From (73)–(75), we obtain that

$$v(q_+) = v(\bar{q}) = -1 + \lambda = (u(0) + \lambda) J_0(\bar{q}), \tag{76}$$

and using (71), (72), (76),

$$\lambda = \frac{u(0) J_0(\bar{q}) + 1}{1 - J_0(\bar{q})} = \frac{r_+^2 - 2r_0^2}{r_+^2} = \frac{r_+^2 - (1 + \bar{u}) R_0^2}{r_+^2}. \tag{77}$$

From (75), (77),

$$r_0 = \varepsilon\bar{q}\Big[\frac{-(1+u(0))J_0(\bar{q})}{2(1-J_0(\bar{q}))}\Big]^{1/2}, \quad \bar{u} = -1-\varepsilon^2\,\frac{\bar{q}^2(1+u(0))J_0(\bar{q})}{R_0^2(1-J_0(\bar{q}))}, \tag{78}$$

and recalling (72), (74), (75), (77), (78),

$$u(r) = \frac{(u(0)+1)J_0(r/\varepsilon) - u(0)\,J_0(\bar{q}) - 1}{1-J_0(\bar{q})}, \quad 0 \le r \le r_+ = \varepsilon\bar{q}, \tag{79}$$

with

$$u(0)+1 = \frac{(1+\bar{u})R_0^2(J_0(\bar{q})-1)}{\varepsilon^2\bar{q}^2 J_0(\bar{q})}. \tag{80}$$

From (77)–(79), we see that there is a one-parameter family of dimple solutions which is uniquely determined by $\bar{u}$, or equivalently by r_0 or by $u(0)$, with $u(0) \in [-1,1]$.

As $u(0) \downarrow -1$,

$$\lambda \uparrow 1, \quad r_0 \downarrow 0, \quad \bar{u} \downarrow -1, \quad \text{and} \quad u(r) \downarrow -1, \quad \forall r \in [0,R_0].$$

As $u(0) \uparrow +1$,

$$\lambda \downarrow \frac{1+J_0(\bar{q})}{1-J_0(\bar{q})}, \quad r_0 \uparrow \varepsilon\bar{q}\Big[\frac{-2J_0(\bar{q})}{2(1-J_0(\bar{q}))}\Big]^{1/2}, \quad \bar{u} \uparrow -1-\varepsilon^2\frac{2\bar{q}^2 J_0(\bar{q})}{R_0^2(1-J_0(\bar{q}))},$$

and $u(r) \uparrow \Big[\frac{2J_0(r/\varepsilon)-J_0(\bar{q})-1}{1-J_0(\bar{q})}\Big]$ for $r \in [0,r_0]$. Note in particular that

$$\lim_{\varepsilon\downarrow 0}\lim_{u(0)\downarrow -1} u(r;\varepsilon,u(0)) = \begin{cases} -1 & \forall r \in (0,R_0], \\ 1 & r = 0, \end{cases} \qquad \lim_{\varepsilon\downarrow 0}\lim_{u(0)\downarrow -1} r_0(\varepsilon,u(0)) = 0.$$

Summarizing the results above,

Theorem 7. *Given $0 < \varepsilon \ll 1$ and $\mathscr{B}_{R_0}(\delta)$, with $R_0 = O(1)$ and $\varepsilon \ll \delta \ll 1$. There exists a unique radial dimple solution, for any $\bar{u} \in (-1, -1-2\varepsilon\bar{q}/R_0)^2 J_0(\bar{q})/(1-J_0(\bar{q}))$, where $\bar{q}$ denotes the first positive zero of $J_1(q)$.*

Finally, using (12) and recalling (3), it is straightforward to verify that

$$E(t) := \frac{1}{\varepsilon|\Omega|}\int_\Omega \{(1-u^2)+\varepsilon^2|\nabla u|^2\}\,dx = \frac{\varepsilon\bar{q}^2}{R_0^2}(1-\lambda^2), \tag{81}$$

where $\bar{q}$ and λ are prescribed in (75) and (77), respectively.

3 Connecting the Dynamic Problems

At least on a somewhat superficial level, the attractor dynamics for both (SD) and (DQOP) can be seen to be similar in $2D$, under the assumptions outlined in Section

2, given $\mathscr{B}_{R_0}(\delta)$ with $0 < \varepsilon \ll \delta \ll 1$ and $R_0 = O(1)$. The steady states for (SD) are given by circular curves centered at the origin, as well as the possible translates of these circular curves that lie within $\mathscr{B}_{R_0}(\delta)$ and the possible nonintersecting union of such curves that lie within $\mathscr{B}_{R_0}(\delta)$. Similarly, the steady states for (DQOP) are given radial solutions which are either annular or dimple solutions, as well as their possible translates within $\mathscr{B}_{R_0}(\delta)$ and solutions obtain via composites of the above given the limitations of the domain $\mathscr{B}_{R_0}(\delta)$. The equivalent mean mass condition for (DQOP) prescribes the mean mass, $\bar{u}$, in terms of an effective radius, r_0, with $0 \leq r_0 < R_0 - \delta$. Accordingly, it is possible to identify a $1-1$ correspondence between the set of radial steady states of (SD), namely circular curves with radius r_0, with $0 < r_0 < R_0 - \delta$, and the set of radial steady states for (DQOP). For simplicity, we may limit our focus to the set of axi-symmetric steady states in both cases, leaving aside for the moment the technical difficulties entailed in taking into account the somewhat larger class of steady states produced by translation. Also, we are neglecting possible radial ringed solutions, composed of concentric circular curves for (DQ) and radially symmetric composite multiple transition (multi-annular solutions, possibly with a dimple solution at the origin).

Thus, if we can identify similar stability properties for both evolutions, we are well on our path to connecting the evolutions. The difficulty arises in considering stability for both evolutions in similar functional analytic settings and in a manner which permits both evolutions to be simultaneously tracked globally in time. We first outline briefly the perhaps easiest and most direct approach, which arises naturally in view of the extant results in the literature for both evolutions, explaining some of the pitfalls in connecting the evolutions. Afterwards, we outline some of the details pertaining to a more robust approach. The more robust approach is based on considering similar minimizing motion evolutionary descriptions for both evolutions, and making a step-by-step connection between the two motions via an appropriate lifting and projecting algorithm.

3.1 Stability

With regard to the stability in the context of (SD) for circular curves, the results of Wheeler [22] are useful. The theory there is based on the following local existence theorem, which is paraphrased below:

Theorem 8. *Suppose that $\Gamma_0 : R \to R^2$ is a periodic regular curve parametrised by arc-length of class $\mathscr{C}^2 \cap W^{2,2}$ with $||\kappa||_2 < \infty$. Then there exists a time $T \in (0,\infty]$ and a unique one-parameter family of immersions $\Gamma : R \times [0,T) \to R^2$ parametrized by arc-length satisfying (SD) such that (i) $\Gamma(0) = \Gamma_0$, (ii) $\Gamma(\cdot,t)$ is of class $\mathscr{C}^\infty$ and periodic of period $\mathscr{L}(\Gamma(\cdot,t))$ for every $t \in (0,T)$, and (iii) T is maximal.*

The regularity requirements on the initial data Γ_0 can be somewhat weakened, [22].

The stability results [22, Theorem 1.1], paraphrased below, are formulated in terms of the normalized oscillation of curvature,

$$\kappa_{osc}(\Gamma(\cdot,t)) := \mathscr{L}(\Gamma(\cdot,t)) \int_\Gamma (\kappa-\bar{\kappa})^2\, ds \quad \text{with} \quad \bar{\kappa}(\Gamma(\cdot,t)) = \mathscr{L}^{-1}(\Gamma(\cdot,t)) \int_\Gamma \kappa\, ds,$$

and the isoperimetric ratio, $\mathscr{I}(\Gamma(\cdot,t)) := \mathscr{L}^2(\Gamma(\cdot,t))\,[4\pi\mathscr{A}(\Gamma(\cdot,t))]^{-1}$, where $\mathscr{A}(\Gamma(\cdot,t))$ denotes the area enclosed by $\Gamma(\cdot,t)$.

Theorem 9. *Suppose that $\Gamma_0 : S^1 \to R^2$ is a regular smooth immersed closed curve with $\mathscr{A}(\Gamma_0) > 0$ and $\int_{\Gamma_0} \kappa\, ds = 2\pi$. There exists a constant $\kappa^* > 0$ such that if*

$$\kappa_{osc}(\Gamma_0) < \kappa^* \quad \text{and} \quad \mathscr{I}(\Gamma_0) < exp\left(\frac{\kappa^*}{8\pi^2}\right),$$

then under (SD) evolution, $\Gamma : S^1 \times [0,T) \to R^2$ with Γ_0 as initial data exists for all time and converges exponentially fast to a round circle with radius $\sqrt{\mathscr{A}(\Gamma_0)/\pi}$.

The results in [22] are quite pleasing. However, one cannot conclude directly from either of these theorems that if $\Gamma_0 \subset \mathscr{B}_{R_0}(\delta)$, then $\Gamma(\cdot,t) \subset \mathscr{B}_{R_0}(\delta)$ for $t \in (0,T)$.

With regard to stability in the (DQOP) context, suppose we wish to demonstrate stability for an annular solution located far from the $\mathscr{B}_{R_0}(\delta)$ boundary. For simplicity, let us consider the stability of an annular solution centered at the origin with $r_0 = O(1)$. We know for (SD) that the encompassed area is maintained, but we do not know off hand that the center of mass does not move. For (DQOP) we similarly know that mass is conserved, but we do not know that the center of mass is time invariant. So, if we are not overly concerned with maintaining the $\mathscr{B}_{R_0}(\delta)$ structure, a reasonable approach is to to consider zero mass perturbations, making use of the H^{-1} gradient structure. Within this context establishing a spectral gap should be straightforward, for zero mass perturbations modulo translations of the center of mass. This would enable us to prove stability of the annular solutions, modulo translation, in analogy with the (SD) results above.

In Section 2.4, there exist $O(\varepsilon)$ energy dimple solutions which equal -1 except on a circular region with $O(\varepsilon^2)$ area. Clearly a "stray" translate of a dimple (or rather a translate of that part of the dimple solution which differs from -1) could be incorporated into the $O(1)$ region where a generic annular solutions equal -1, with a small alteration in the radii of the annular solution to accommodate the additional mass. Such a small "droplet-like" perturbation would constitute a small energy perturbation, though not covered by the discussion above, as they are not zero mass perturbations. As such perturbations are natural to consider, we remark here that it is possible to construct a sequence of energy lowering and mass preserving perturbations, which allow the dimple to lose height and to transfer away volume (mass). Details to be published elsewhere, together with the (DQOP) stability results described above.

3.2 Minimizing motions

Minimizing motion evolution formulation for (SD). In Fonseca et.al. [11], De Giorgi's minimizing movement approach is implemented within a framework with H^{-1} gradient flow structure, to prove short time existence, uniqueness, and regularity for the motion of an elastic thin film which evolves by anisotropic surface diffusion. Their approach yields, as a subcase, a proof of short time existence, uniqueness, and regularity for a spatially period $1D$ curve prescribable by the graph of a function, $\Gamma = \{(x, h(x) : 0 < x < b\}$, with $b \in (0, \infty)$.

More specifically, starting with b periodic data initial, $h_0 \in H^2_{loc}(R)$, a sequence of approximants $h_{i,N}$ are defined inductively, for $T > 0$, $N \in \mathscr{N}$, $i = 1, \ldots, N$, as a minimizer of $E(h) + \frac{1}{2\tau} d^2(h, h_{i-1,N})$, where $E(h)$ denotes the system energy, $\tau = T/N$, and d measures the H^{-1} distance between h and $h_{i-1,N}$. Their choice for d is based on the following H^{-1} norm for curves Γ,

$$||f||_{H^{-1}(\Gamma)} := \sup_{||\psi||_{H^1(\Gamma)=1}} \int_\Gamma f \psi d\mathscr{H}^1(z),$$

which can be expressed as

$$||f||^2_{H^{-1}(\Gamma)} = \int_\Gamma \left(\Phi(z) - \frac{1}{|\Gamma|} \int_\Gamma \Phi d\mathscr{H}^1 \right)^2 d\mathscr{H}^1(z) + \left(\int_\Gamma f \, d\mathscr{H}^1 \right)^2,$$

where $\Phi(z) := \int_{\Gamma(z_0,z)} f(w) \, d\mathscr{H}^1(w)$, $z_0 = (0, h_{i-1,N}(0))$ and $\Gamma(z_0, z)$ denotes the arc of Γ connecting z_0 with z. Accordingly they base their scheme on the $(H^{-1})^2$ penalization

$$\int_\Gamma \left(\int_{\Gamma(z_0,z)} f(w) \, d\mathscr{H}^1(w) \right)^2 d\mathscr{H}^1(z), \tag{82}$$

for $f = h_i - h_{i-1}$, with two constraints reflecting zero mean and periodicity:

$$\int_\Gamma f \, d\mathscr{H}^1 = 0, \quad \int_\Gamma \int_{\Gamma(z_0,z)} f(w) \, d\mathscr{H}^1(w) \, d\mathscr{H}^1(z) = 0. \tag{83}$$

To implement their approach in our context, an extension of their approach is needed for closed imbedded planar curves, Γ. The resultant penalization with constraints is similar to (82),(83), but incorporates an orientational weight.

Minimizing motion evolution formulation for (DQOP). In [14] a minimizing movement scheme is developed for proving the existence of weak solutions for a class of degenerate parabolic equations of fourth order, which includes (DQOP) given the assumptions outlined in Section 1. This class of evolution equations are shown to correspond to a gradient flow with respect to a Wasserstein-like transport metric, W_m, and the weak solutions are obtained via a scheme based on curves of maximal slope.

The metric, W_m, which was shown in [9] to constitute a genuine metric, is neither the L^2-Wasserstein distance not a flat Hilbertian metric; rather it corresponds to a metric tensor based on the following construction: given a tangential vector $v := \partial_s \rho(0)$ to a smooth curve $\rho : (-\varepsilon, \varepsilon) \to L^1(\Omega)$ of strictly positive densities $\rho(s)$ at $\rho_0 = \rho(0)$, it assigns the length

$$||v||^2 = \int_\Omega |D\psi(x)|^2 m(\rho_0(x))\,dx, \text{ with } -\nabla\cdot(m(\rho_0(x))\nabla\psi(x)) = v, \quad x \in \Omega, \tag{84}$$

with variational boundary conditions on $\partial\Omega$. Within the context of our framework, $\Omega = \mathscr{B}_{R_0}(\delta)$, as discussed in Section 1, and the ambient space for the scheme is the metric space $(X(\Omega), W_m)$ where

$$X(\Omega) := \Big\{ u \in L^1(\Omega) \,|\, -1 \le u \le 1 \quad a.e. \quad x \in \Omega, \quad \int_\Omega u\,dx = \bar{u} \Big\}.$$

The minimizing movement scheme starts with initial conditions $u_0 \in X(\Omega)$ such that $E[u_0] < \infty$, and approximants are defined by setting

$$u_\tau^0 := u_0, \quad u_\tau^{n+1} : -\arg\min \Phi_\tau^n \in X(\Omega), \quad \Phi_\tau^n(v) := \frac{1}{2\tau} W_m(u_\tau^n, v)^2 + E[v],$$

with $E[u] := \infty$ for $u \notin H^1(\Omega)$. Piecewise constant interpolation is then used to define an approximation $\tilde{u}_\tau(t) : [0,\infty) \to X(\Omega)$.

Lifting and projecting: a Hilbert expansion approach. As we have seen, there exist minimizing motion schemes for both (SD) and (DQOP). To establish a rigorous connection between the two evolutions, (SD) and (DQOP), it makes sense to consider similar time steps, $\tau = T/N$, for both minimizing motions. It is also reasonable to consider similar initial conditions, perhaps similarly perturbed steady state solutions to (SD) and (DQOP); for simplicity, we might consider a circle and an annular solution, respectively, with equivalent mean mass and both centered at the origin. We want to compare the results of the minimizing motion schemes and to demonstrate that they yield the same motion in the limit as $\varepsilon \to 0$ and $\tau \to 0$. Here the difficulty is that at each time step the motion for (SD) is describable in terms of curves Γ_t belonging to $\mathscr{M}$, the set of smooth simple closed curves, and the motion for (DQOP) is described in terms of functions $u(\cdot,t)$ defined for all $x \in \mathscr{B}_{R_0}(\delta)$. Modulo issues of regularity and structure, the zero level set of $u(\cdot,t)$ yields a "projection" of $u(\cdot,t)$ onto the set of curves $\mathscr{M}$. The matter of "lifting" Γ_t to obtain $u(\cdot,t)$ is more delicate. In the context of connecting the Cahn-Hilliard equation with the Mullins-Sekerka problem, Carlen, Carvalho & Orlandi [4] used a Hilbert expansion approach to construct a globally defined function $u(\cdot,t)$ from a curve $\Gamma_t \in \mathscr{M}$. They constructed an approximation for $u(\cdot,t)$ based on three types of terms, (i) terms depending on Γ_t, (ii) terms reflecting local corrections near Γ_t, and (iii) terms reflecting long range corrections.

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